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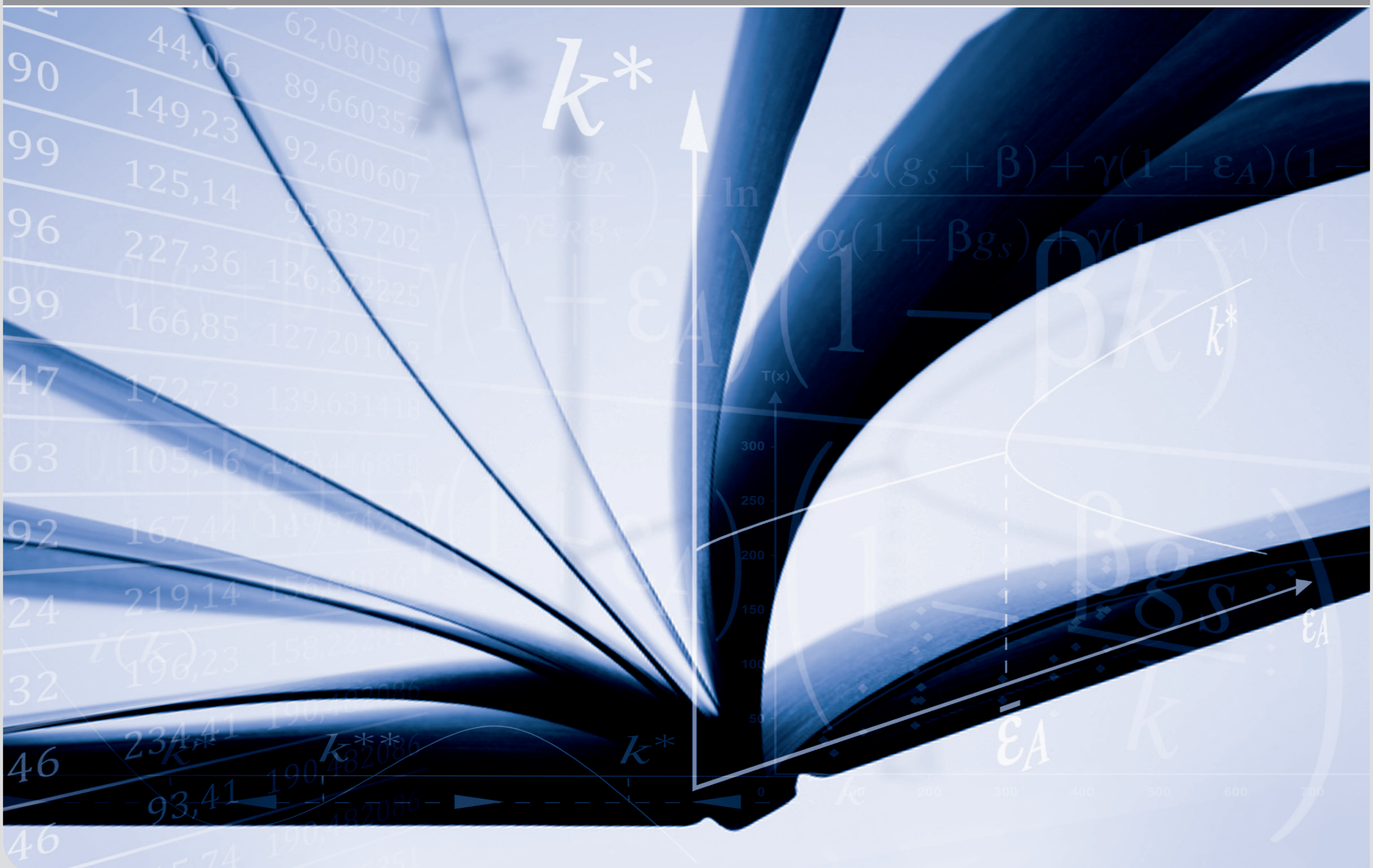

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Condorcet Solutions in Frugal Models of Budget Allocation*

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Abstract

We study a voting model with incomplete information in which the evaluation of social welfare must be based on information about agents' top choices plus general qualitative background conditions on preferences. The former is elicited individually, while the latter is not. We apply this 'frugal aggregation' model to multi-dimensional budget allocation problems, relying on the specific assumptions of convexity and separability of preferences.

We propose a solution concept of ex-ante Condorcet winners which is widely and flexibly applicable and naturally incorporates the epistemic assumptions of particular frugal aggregation models. We show that for the case of convex preferences, the ex-ante Condorcet approach naturally leads to a refinement of the Tukey median. By contrast, in the case of separably convex preferences, the same approach leads to different solution, the 1-median, i.e. the minimization of the sum of the L_1-distances to the agents' tops. An algorithmic characterization renders the latter solution analytically tractable and efficiently computable.

Keywords: Social choice under partial information; frugal aggregation; ex-ante Condorcet approach; participatory budgeting; Tukey median.

JEL classification: D71

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1 Introduction

Many economic and political decisions involve the allocation of resources under a budget constraint. Examples are the allocation of public goods, the redistribution across classes of beneficiaries, the allocation of tax burden, the choice of intertemporal expenditure streams, or the macro-allocation between expenditure, tax receipts and net debt. Here we explore the possibility of taking these collective decisions by voting.

Standard approaches to preference aggregation and voting assume ordinal or even cardinal preference information as their input. Their application to public resource allocation problems poses substantial difficulties for a variety of reasons. First, at the foundational level, except for the one-dimensional case with two public goods and single-peaked preferences (Black, 1948; Arrow, 1951/63), one is faced with generic impossibility results under almost every reasonable domain restriction (Kalai *et al.*, 1979; Le Breton and Weymark, 2011) just as in spatial voting models (Plott, 1967). In particular, in higher dimensions there is no hope to generally find a Condorcet winner even if all agents have well-behaved preferences. Indeed, the indeterminacy of majority voting is generic and can be severe; for example, generically every alternative can be the outcome of a dynamic (non-strategic) majority vote for an appropriate agenda (McKelvey, 1979). Thus, from the general point of view of ordinal social choice theory, it is even conceptually unclear what allocations an optimal voting rule should aim at. Second, and especially important for the voting as opposed to the abstract social choice perspective, at a more pragmatic level, a basic problem already arises from the sheer number of alternatives which grows exponentially in the number of dimensions (i.e. alternative uses of the public resource). Collecting a complete ordering over the set of all alternatives from each agent (whether citizen or representative) as required for many voting mechanisms is often simply infeasible. Clearly, much is to be said for making the task of the voter as easy as possible.

Here, we take a minimalist approach by assuming that only voters' preference tops are individually elicited. (Note that one needs to know at least voters' tops to have an adequate basis for collective decisions in the most straightforward case in which voters agree unanimously). We also allow the social evaluator to rely on some background information about the structure of preferences. As in other economic settings, convexity of preferences s frequently highly plausible. Additional assumptions such as preference separability may also be plausible and useful. This background information is represented as a set of possible preference orderings, or 'models.'

This paper aims to determine which of the feasible social choices (allocations) is normatively best in light of the elicited and background information. To address this question, we take a *qualitative, non-probabilistic approach* which assumes that this information is known *and nothing else*. Formally, the epistemic state of the social evaluator is modeled by a set of possible preference profiles. An obvious methodological alternative would be to model the uncertainty in a Bayesian manner by assigning probabilities to the profiles in this set. But, for multiple reasons, such an approach appears to have limited appeal here. In particular, *whose* subjective probability is supposed to be the basis of the evaluation? If the social evaluator was understood as a social planner ('bureaucrat'), one may think of the required judgmental input as reflecting the planner's expertise; but in a voting context, the social evaluator is naturally viewed as 'the group' at a constitutional stage at which individual preference profiles are unknown. A qualitative specification of the epistemic basis for the social evaluation seems especially attractive since it would appear to be much more amenable for agreement at the constitutional stage.

Alluding to the notion of 'fast and frugal heuristics' due to Gigerenzer and Goldstein (1996), we call our approach 'frugal' to mark its reliance on minimally demanding informational assumptions as well as on a coarse, qualitative treatment of uncertainty. Our aim is to show that even from these minimalist, pragmatically appealing premises, attractive and credible choice implications can be derived. We formulated the search for a frugal normative optimum as the combination of two notoriously difficult rational choice problems: The search for a frugal normative optimum combines two notoriously difficult rational choice problems: the Arrowian problem of interpersonal aggregation of preference rankings in the absence of interpersonal comparisons, and the problem of rational individual choice under non-probabilistic ignorance.[1] As both problems are prone to 'impossibility' results, it would appear that the combination of the two must be an even tougher nut to crack.

As plausible as this expectation of yet more and deeper impossibilities may seem, we show how they can be overcome by means of a (novel) 'ex-ante Condorcet' (EAC) approach for suitable modeling choices. The EAC approach is based on ex-ante comparisons between pairs of alternatives based on the interval of possible preference vote counts. A basic yet fundamental observation yields a canonical ex-ante majority relation which can be defined independently of subjective attitudes such as pessimism, optimism, ambiguity-aversion or ambiguity-seeking. The EAC approach selects – whenever possible – the maximal elements of this relation, the ex-ante Condorcet winners. In the models at the center of this paper, we show that such winners do exist (either outright or with appropriate qualification) and characterize them.

In the simplest case of public resource allocation among two possible uses, convexity is very powerful as it implies single-peakedness of ex-post preferences. The ex-post Condorcet winner is the median of voters' tops and thus known ex-ante and equal to the ex-ante Condorcet winner. Yet, with more than two goods, things get a lot more complicated. For instance, the 'plain convex model' in which knowledge of convexity alone is assumed, does not yield very useful implications. Indeed, generically (i.e. with tops in general position), all tops are ex-ante Condorcet winners when there are more than two goods, just like in the absence of any preference information whatsoever (see Proposition 4). Admitting all convex preferences on equally possible footing turns out to be too radical an assumption to leverage the obvious potential of the knowledge of convexity. To achieve this, we refine the plain convex model in two different ways.

The first such approach 'regularizes' the plain convex model by imposing restrictions on the pattern of ex-post preferences given a profile of tops. To execute this formally, we assume a parametric form of convex preferences, namely quadratic preferences. A particular quadratic form $\mathcal{Q}$ describes the substitution-complementation-separability 'structure' of a quadratic preference ordering in terms of the cross-partials of the utility function. 'Regularity' in the relevant sense is introduced by assuming that the ex-post distribution of voters' preference

[1] For the latter, see the classical reference by Luce and Raiffa (1957), as well as Nehring (2000, 2009) for a more recent perspective.

structures $\mathcal{Q}$ is stochastically independent of the distribution of their tops. This assumption defines the 'homogeneous quadratic model.' It holds in particular whenever all voters share the same $\mathcal{Q}$; such profiles are instances of intermediate preferences à la Grandmont (1978).

In the second approach, the social evaluator assumes that ('knows that') individual voters' preferences are separable besides being convex, but assumes nothing further about ex-post profiles of preferences. Separability seems often appealing for broad expenditure categories, but may make less sense for more finely described goods.

The two main results of the paper characterize the ex-ante Condorcet winners in the two models – in the separably convex case by adding a localization twist. In the homogeneous quadratic model, we show in Theorem 1 that an ex-ante Condorcet winner always exists and is always a *Tukey median.* Tukey medians are classical coordinate-free generalizations of ordinary medians to multiple dimensions, see Small (1990) for a classic survey and Rousseeuw and Hubert (2017) for comprehensive treatment. Conversely, a Tukey median is an ex-ante Condorcet winner whenever it is 'strict' (as formally defined in Section 3). In a variant of the first main result, we also provide an alternative characterization of strict Tukey medians as ex-ante Condorcet winners based on a 'similarity hypothesis' that is weaker in effect and is imposed directly on the range of 'plausible' majority margins in each binary comparison (Theorem 2).

In the separably convex model, unrestricted ex-ante Condorcet winners need not always exist, but local ex-ante Condorcet winners do exist and are singled out by appeal to the underlying separability assumption on preferences. These local ex-ante Condorcet winners are characterized in a number of ways. The first characterization in Theorem 3 shows them as equivalent to metric medians based on the L_1-norm ('1-medians').[2] In the present setting, 1-medians have a natural resource allocation interpretation and permit a highly informative and operationally transparent characterization that ensures fast spread-sheet computability even in high dimensions (Theorem 4). As a corollary, Theorem 4 yields a characterization of 1-medians as a refinement of a coordinate-based version of the Tukey median (Proposition 9).

Related Literature

To the best of our knowledge, the present is the first 'frugal aggregation' model of its kind. But there are, of course, related approaches in the literature. The Tukey median has been studied implicitly in the social choice literature inasmuch as it is equivalent to the outcome of the minimax voting rule in standard spatial voting with Euclidean preferences (Kramer, 1977; Demange, 1982; Caplin and Nalebuff, 1988). In fact, the latter two contributions amount to substantive analyses of properties of the Tukey median itself which are directly relevant to the present study (see Sections 3 and 5). The Euclidean model can be viewed as a special, degenerate case of a frugal model in which voters preferences conditional on their top are known. It is intuitive and supported by heuristic argument in Section 3.4 below that the (Condorcetian) normative frugal optimum in such a model is the L_2-median (which minimizes

[2] The 1-medians can be viewed as an instance of the median rule known from general aggregation theory, see, e.g., Barthélémy and Monjardet (1981); Nehring and Pivato (2021). Our terminology deviates from some of the statistical literature which denotes by '1-median' the minimizers of the Euclidean (i.e. L_2-)distance, see e.g., Vardi and Zhang (2000).

the sum of Euclidean distances), not the Tukey median. Methodologically, it seems arguable that the great popularity of the spatial voting model stems in no small part from the fact that preferences are determined by voters tops (rather than the appeal of the assumption of Euclidean preferences per se). The proposed frugal approach may thus also be attractive from a more analytical rather than strictly normative approach by explicitly modeling the analysts' absence of knowledge of the voters' precise preferences.

Most work of theoretical interest in the frugal theme of this paper has come from the computer science literature, see Boutilier and Rosenschein (2016) for an overview.[3] One strand explores the implications of partial knowledge of complete (ex-post) preference profiles for inferences about the outcome of standard social choice rules and criteria, e.g. via the notions of 'possible' vs. 'necessary' winners (Konczak and Lang, 2005); in this vein, we characterize the possible Condorcet winners in our EAC approach (see Proposition 1 below). Another (smaller) strand in the literature adopts a decision-theoretic ex-ante approach as this paper does. Some papers seek solutions that maximize expected welfare based on some utilitarian welfare criterion and a probability distribution over profiles, frequently uniform. Others argue for the modeling of the social evaluator's epistemic state in terms of a set of possible profiles, as we do, and argue for the application of classical criteria of decision making under ignorance such as maximin or minimax regret (Lu and Boutilier, 2011). In the highly complex state spaces associated with the epistemic models studied here, it may be very difficult to execute these approaches if that is possible at all. Significantly, the two quoted strands share the major conceptual limitation of having to rely on an interprofile-comparable standard of aggregate welfare ex-post. Thus, they in fact assume that the Arrovian problems of coherent aggregation and interpersonal non-comparability have been solved or assumed away, e.g. by assuming strong forms of utilitarian aggregation ex-post.[4]

By contrast, the EAC approach introduced here rests on an evaluation of decisions in pairs of alternatives taking the full state space (set of possible profiles) into account. In such pairwise comparisons, the majority criterion carries over naturally to the ex-ante stage, without raising new issues of interpersonal comparison, and allowing a tractable characterization in many cases. These pairwise comparisons need then be put together to obtain a coherent rationale for an ex-ante evaluation of complex choices such as budget allocations. At this juncture, Arrovian style issues of coherent aggregation might arise in principle. It is a rather remarkable finding of this paper that, in the the frugal models studied here, these problems do not arise or are resolved easily.

With respect to the focal application to the allocation of public budgets, there is an important recent literature on 'participatory budgeting' with intended application to cities and local communities (Shah, 2007). Participatory budgeting schemes have been put into practice at various scales in many places around the world including Porto Allegre (Brasil), Paris, Barcelona, New York City, and at various other places. The ballots are typically very parsimonious, often taking the form of a set of projects approved.[5] Again, most of

[3]We thank Jérôme Lang who pointed us to the pertinent literature.

[4]We use the ex-ante vs. ex-post distinction purely for conceptual purposes, without any assumption of an ex-post stage at which the actual profile of preferences is observed.

[5]See, e.g., the open source project 'Stanford Participatory Budgeting Platform' (https://pbstanford.org) which offers guidance and allows municipalities, cities and other institutions to run participatory budgeting

the theoretical contributions come from the computer science community, with a focus on indivisibilities and on 'proportionality' considerations to ensure that the interest of different local subcommunities are fairly represented (Aziz and Shah, 2020). By contrast, our focus is on divisible budgets (continuous or discrete), and on finding allocations that best satisfy the aggregate interest (in parallel with most of standard voting theory).[6]

To the best of our knowledge, none of the contributions take an explicitly frugal normative approach. Instead, they either focus on the voting mechanism directly, or make strong and specific assumptions on preferences. Interestingly, the two recent contributions (Goel *et al.*, 2019; Freeman *et al.*, 2021) assume L_1-preferences, and show that under this domain restriction there exist strategy-proof selections from the 1-median. The separably convex model analyzed in Section 4 below can be viewed as providing a foundation of sorts for employing such preferences as a focal case in that normatively, the EAC solution can be interpreted as the utilitarian optimum relative to imputed L_1-preferences.

A special case of the discrete separably convex model with contact to a different strand of the literature is the committee problem discussed in Section 4.6 below, see in particular (Lang and Xia, 2016).

Plan of Paper

In the next section, Section 2, we introduce the general EAC approach. The subsequent sections address the budget allocation problem, Section 3 under the background assumption of convex preferences, Section 4 under the assumption of convexity plus separability. Section 3.1 considers the 'plain' convex model and shows that the EAC solution in this model coincides with generic plurality rule. Section 3.2 introduces the model of quadratic preferences and shows that under a suitable homogeneity condition on profiles of ex-post preferences the EAC solution coincides with (a refinement of) the Tukey median. Section 3.3 shows that the homogeneity assumption in the quadratic model can be replaced by a (a priori weaker) 'similarity' condition imposed directly on the possible majority intervals. Section 3.4 discusses properties of the Tukey median, foremost its affine equivariance, and compares it to potential alternatives, in particular the 'p-medians' that are based on the minimization of aggregate L_p-distances.

Section 4 introduces the 'separably convex' model. The basic facts are gathered in Sections 4.1-4.4; the (localized) EAC solution is shown to coincide with the 1-median in Section 4.5. The special case of the committee selection problem is treated in Section 4.6. The efficient computability of the EAC solution is demonstrated in Section 4.7. An interconnection of the EAC solution of the convex model (the strict Tukey median) and the separably convex model (the 1-median) is provided in Section 4.8. Section 4.9 shows how the EAC approach can be applied to the case of indivisible projects.

With reference to the work by Caplin and Nalebuff (1988, 1991), Section 5 points out that in certain situations pragmatic considerations might favor the adaption of the 1-median rather

elections online.

[6] In Section 4.9 below, we show how to accommodate indivisibilities in individual projects in the presence of divisible 'general expenditure.'

than Tukey median as an actual voting rule even when the epistemic case for separability is weak. Section 6 concludes.

An extensive appendix provides further illustrations, additional results and extensions (Appendix A.1-A.8). All remaining proofs are gathered in Appendix B.

2 Condorcet Winners, Ex-Ante

We envisage a social evaluator who has to choose from a universe of alternatives X, based on epistemic states of a specific structure: a profile of individual top alternatives $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, for some number $n \in \mathbb{N}$ of *voters*, and a corresponding set of possible ex-post preferences. The later is described by a set of 'admissible' profiles $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ of ex-post preferences $\boldsymbol{\mathcal{M}}$. We allow for any finite size of the electorate; hence if we denote by $\mathcal{R}$ the set of all weak orders on X, we have $\boldsymbol{\mathcal{M}} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{R}^n$. In the following, we will refer to $\boldsymbol{\mathcal{M}}$ as a **model (of preferences)**. It specifies the background assumption on the qualitative structure of ex-post preferences given a profile of tops.

For simplicity we assume that every admissible preference $\succcurlyeq_i$ has a unique top alternative in X which we denote by $\tau(\succcurlyeq_i)$. Given a model $\boldsymbol{\mathcal{M}}$ and a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, we denote the epistemic state of the social evaluator by $\Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$, i.e.

$$\Omega_{((\theta_1,...,\theta_n),\boldsymbol{\mathcal{M}})} \;:=\; \{(\succcurlyeq_1, ..., \succcurlyeq_n) \in \boldsymbol{\mathcal{M}} \mid \theta_i = \tau(\succcurlyeq_i) \text{ for all } i\}\,.$$

For all distinct $x, y \in X$, an epistemic state $\Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ induces an interval $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)$ in the possible support counts in the vote of x against y, specifically let

$$m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) \;:=\; \Big[m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)\,,\, m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)\Big]\,,$$

where

$$m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) \;:=\; \min_{\succcurlyeq\,\in\Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}} \#\{i : x \succ_i y\}, \tag{2.1}$$

$$m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) \;:=\; \max_{\succcurlyeq\,\in\Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}} \#\{i : x \succ_i y\}. \tag{2.2}$$

The family of these intervals will be what matters in our analysis. Sometimes a submodel $\boldsymbol{\mathcal{M}}' \subseteq \boldsymbol{\mathcal{M}}$ induces exactly the same support count intervals as the model $\boldsymbol{\mathcal{M}}$ itself, for all θ and all $x, y \in X$; in that case, we call $\boldsymbol{\mathcal{M}}'$ a *rich submodel* of $\boldsymbol{\mathcal{M}}$.

In deciding ex-ante on a hypothetical choice between x and y, it is natural to base this choice on a comparison of the intervals $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)$ and $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x)$. A definitive and unambiguous comparison based on knowledge alone is possible if and only if the support intervals $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)$ and $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x)$ do not overlap, i.e. if it is *known* that more voters prefer x to y than vice versa – whatever the precise margin may be. If $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)$ lies entirely above $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x)$, we say that x is a *necessary majority winner over* y, and denote this by

$$xP^{\text{nec}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}y \;:\Longleftrightarrow\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) > m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x).$$

The maximal elements with respect to this relation is referred to as the **majority admissible set**, and denoted by

$$\mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \ := \ \{x \in X \mid \text{ for no } y \in X, \ yP^{\mathrm{nec}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x\}.$$

An aggregation rule is called *majority admissible* if it only chooses from the majority admissible set whenever that set is non-empty. From the definition it is immediate that the majority admissible set is increasing with the underlying model, i.e., for all $\boldsymbol{\theta}$,

$$\boldsymbol{\mathcal{M}} \subseteq \boldsymbol{\mathcal{M}}' \implies \mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \subseteq \mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}') \tag{2.3}$$

(with equality on the right hand side if $\boldsymbol{\mathcal{M}}$ is a rich submodel of $\boldsymbol{\mathcal{M}}'$). Moreover, it is evident that every Condorcet winner with respect to any profile of ex-post preferences is majority admissible. Under a weak richness condition on the model, the converse holds as well, i.e. the majority admissible set coincides with the possible ex-post Condorcet winners. Specifically, say that a model $\boldsymbol{\mathcal{M}}$ is *copious* if, for all $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and $x \in X$ the following condition is satisfied. For all $J \subseteq \{1, ..., n\}$ and all $Y \subseteq X$,

$$\forall i \in J \ \forall y \in Y \ \exists \succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}, \ y \succ_i x \implies \exists \succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})} \forall i \in J \ \forall y \in Y, \ y \succ_i x. \tag{2.4}$$

An alternative x is an *(ex-post) Condorcet winner* at a profile $\succcurlyeq$ if for no alternative $y \in X$, $\#\{i : y \succ_i x\} > \#\{i : x \succ_i y\}$, i.e. if no alternative receives strictly higher support than x in a binary comparison.

Proposition 1. *Suppose that $\boldsymbol{\mathcal{M}}$ is copious. For all $x \in X$ and all profiles $\boldsymbol{\theta}$, $x \in \mathrm{MA}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ if and only if there exists a profile $\succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ such that x is a Condorcet winner at $\succcurlyeq$.*

(Proof in appendix.)

With sufficient ignorance, the relation $P^{\mathrm{nec}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ will tend to be incomplete. In those cases, a substantive ex-ante comparison requires a balance of uncertainties, i.e. a comparison of the intervals $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y)$ and $m_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x)$ when they overlap. Due to the 'complementarity' of vote counts ex-post, a comparison of the lower and upper endpoints of these intervals must yield the same result; formally we have:

Fact 2.1. *For all $\boldsymbol{\theta}$ and all distinct $x, y \in X$,*

$$m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) \geq m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x) \iff m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, y) \geq m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y, x).$$

To verify this, simply observe that, for all distinct $w, z \in X$, $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(w, z) + m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(z, w) = n - n_0$, where n_0 is the number of voters who are indifferent between w and z at all profiles in $\Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$.

Hence an *unambiguous* balance of uncertainties ex-ante is possible; in contrast to the classical theory of decision making under ignorance (see, e.g. the survey Luce and Raiffa, 1957), there is no need or even meaningful role for an evaluators degree of pessimism vs. optimism (ambiguity-aversion vs. ambiguity-seeking in more modern terminology).

Define the **ex-ante majority relation** $R_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ as follows. For all distinct $x, y \in X$,

$$\begin{aligned} xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}y \quad &:\Longleftrightarrow \quad m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y) \geq m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x) \\ &\Longleftrightarrow \quad m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y) \geq m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x). \end{aligned} \tag{2.5}$$

The maximal elements with respect to the ex-ante majority relation are referred to as the **ex-ante Condorcet winners**, i.e.

$$\mathrm{CW}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}) \ := \ \{x \in X \mid xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}y \text{ for all } y \in X\}.$$

Evidently, $P^{\mathrm{nec}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})} \subseteq P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ where $P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ is the asymmetric part of the ex-ante majority relation $R_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$, and hence,

$$\mathrm{CW}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}) \ \subseteq \ \mathrm{MA}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}).$$

An aggregation rule is called **ex-ante Condorcet consistent** if it selects all ex-ante Condorcet winners (if there are any). Observe that every rich submodel $\boldsymbol{\mathcal{M}}' \subseteq \boldsymbol{\mathcal{M}}$ induces the same majority admissible set and the same set of ex-ante Condorcet winners, i.e. for all profiles $\boldsymbol{\theta}$, $\mathrm{MA}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}') = \mathrm{MA}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})$ and $\mathrm{CW}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}') = \mathrm{CW}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})$.

Before proceeding, it is worth noting the generality of the ex-ante Condorcet approach. Here we describe the epistemic state of the social evaluator in terms of a set of possible ex-post preferences. Alternatively, the epistemic state of the social evaluator might be given in terms of precise probabilities, or, as a joint generalization of both, in terms of sets of priors. In the first case, the role of the majority interval would be desribed by an estimated majority (support count); in the latter case, the majority interval would be generalized as an interval of expectations, but again, an unambiguous balancing of uncertainties in the manner described in Fact 2.1 is available. In Section 3.3, we will characterize ex-ante Condorcet winners in a reduced-form approach in which 'plausible' majority intervals are specified directly.

3 Budget Allocation: Convex Preference Models

In the rest of this paper, we will study the following *budget allocation problem.* Suppose that a group of agents has to collectively decide on how to allocate a fixed budget $Q \geq 0$ to a number L of public goods ('projects'). Throughout we assume fixed prices, thus the problem is fully determined by specifying the expenditure shares. Furthermore, we assume that all individuals have monotone preferences. Expenditure x^ℓ on public good ℓ may be bounded from below and above, so that feasibility requires $x^\ell \in [q^\ell_-, q^\ell_+]$ for some integers q^ℓ_-, q^ℓ_+ where we allow that $q^\ell_- = -\infty$ and/or $q^\ell_+ = \infty$. Together, these assumptions allow us to model the allocation problem as the choice of an element of the following $(L-1)$-dimensional polytope

$$X := \left\{ x \in \mathbb{R}^L \mid \sum_{\ell=1}^{L} x^\ell = Q \text{ and } x^\ell \in [q^\ell_-, q^\ell_+] \text{ for all } \ell = 1, ..., L \right\}, \tag{3.1}$$

where $x = (x^1, ..., x^L)$. The space X is referred to as the set of *feasible allocations*, or alternatively, in our context as a *resource agenda.*

A weak order $\succcurlyeq$ on X is *convex* if, (i) for all $x, y, z, w \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 \leq t \leq 1$, $x \succcurlyeq w$ and $z \succcurlyeq w$ jointly imply $y \succcurlyeq w$. It will be convenient to also require the following property: (ii) for all $x, y, z \in X$, $y = t \cdot x + (1-t) \cdot z$ for some $0 < t < 1$, and $x \succ z$ jointly imply $x \succ y$. Observe that this is still weaker than the standard textbook notion of *strict convexity*; for instance, linear preferences satisfy both conditions (i) and (ii) while they are usually not considered as 'strictly' convex. We will maintain both assumptions throughout and denote the set of all weak orders satisfying (i) and (ii) by $\mathcal{R}_{\text{co}}$.

3.1 The Plain Convex Model

The most straightforward starting point under the general assumption of convexity of preferences is to consider the model $\boldsymbol{\mathcal{M}}_{\text{co}} := \cup_{n \in \mathbb{N}} (\mathcal{R}_{\text{co}})^n$, which we refer to as the **plain convex model**.

3.1.1 The One-Dimensional Case: Median Voting

In the case of two goods, i.e. $L = 2$, $\mathcal{R}_{\text{co}}$ coincides with the set of all single-peaked preferences on $X \subseteq \mathbb{R}$, and the choice of the median top(s) constitutes the unique ex-ante Condorcet consistent aggregation rule; specifically, we have the following result. For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, denote by θ_{med} the unique median if n is odd, and by $[\theta_{\text{med}^-}, \theta_{\text{med}^+}]$ the median interval if the number of voters is even. The following result is easily verified.

Proposition 2. *Suppose that $L = 2$, and let $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ be a profile of tops in X. Then,*

$$\text{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{co}}) \;=\; \text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{co}}) \;=\; \begin{cases} \{\theta_{\text{med}}\} & \textit{if } n \textit{ is odd} \\ [\theta_{\text{med}^-}, \theta_{\text{med}^+}] & \textit{if } n \textit{ is even} \end{cases}.$$

Thus, in the one-dimensional case the ex-post and ex-ante Condorcet criterion give the same result under single-peakedness. The reason is, of course, that under knowledge of single-peakedness any given top uniquely determines the preference on both sides of the top, and that is all what is needed to apply the Condorcet criterion.[7]

3.1.2 The Multi-Dimensional Case: Generic Plurality Rule

In the multi-dimensional case, a result similar to Proposition 2 holds if the top profile is contained in a one-dimensional subspace;[8] but in general, majority admissibility has very weak implications in the plain convex model. Specifically, say that a set $Y \subseteq X$ is *collinear* if the points in Y all lie on the same line; furthermore, say that $Y \subseteq X$ is *in general position* if no three distinct elements of Y are collinear.

Proposition 3. *Let $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ be a profile of tops; then $x \in \text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{co}})$ if and only if, for no subset $J \subseteq \{1, ..., n\}$ of more than $n/2$ voters, the set $\{x\} \cup \{\theta_i\}_{i \in J}$ is collinear and $x \notin co(\{\theta_i\}_{i \in J})$.*

[7] Note that our somewhat stronger condition on preferences formulated above guarantees that preferences are strictly decreasing on both sides of the top; the only remaining uncertainty in the one-dimensional case is about the comparison of alternatives from different sides of the top.

[8] see, Fact B.1 in the appendix.

(Proof in appendix.)

Frequently, it will be useful to identify profiles of individual tops with *type profiles* of tops with different popular mass. Specifically, we denote by $\boldsymbol{\theta} = (\theta_1; p_1, ..., \theta_m; p_m)$ the (anonymous) profile in which the fraction p_i of all voters has top θ_i, where $0 < p_i \leq 1$ and $\sum_i p_i = 1$; in that context, we also refer to θ_i as the *type* of voter i, and we assume without of loss of generality that the θ_i are pairwise distinct.

In the plain convex model, the ex-ante Condorcet winner coincides with the plurality winner generically, as follows.

Proposition 4. *Consider a type profile $(\theta_1; p_1, ..., \theta_m; p_m)$ such that the set $\{\theta_1, ..., \theta_m\} \subseteq X$ is in general position. If p_{i^*} is maximal among $\{p_1, ..., p_m\}$, then $\theta_{i^*} \in \mathrm{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{co}})$. Moreover, if p_{i^*} is uniquely maximal among $\{p_1, ..., p_m\}$, then*

$$\mathrm{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{co}}) \;=\; \{\theta_{i^*}\}.$$

(Proof in appendix.)

This is somewhat paradoxical. Intuitively it would appear that preference convexity contains substantial information beyond knowledge of the tops which Proposition 4 appears to contradict. What is amiss?

Example 1. *Consider a large set of voters with pairwise distinct tops in an ε–neighborhood U of, say the point $(1, 1, 1)$, in general position. In addition, suppose that two voters are concentrated on one point outside that neighborhood, say at $x = (0, 0, 3)$ (see Figure 1). Then, according to Proposition 4, x is the unique ex-ante Condorcet winner.*

For example, we have $xP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{co}})}y$ where $y = (\delta, \delta, 3 - 2\delta)$ for sufficiently small $\delta > 0$. Indeed, $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{co}})}(x, y) = 2$ while $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{co}})}(y, x) \leq 1$ (cf. Fig. 1); note that, if all tops plus the point y are in general position, it is even possible that all *voters with top in U prefer x to y in the plain convex model.*

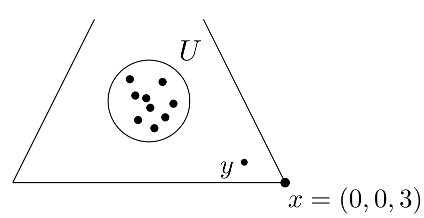

Figure 1: Illustration of Proposition 4

In any case, we obtain $m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{co}})}(x, y) \geq n - 1$, i.e. there is almost complete ignorance about the support count ex-ante. If the epistemic state of the social evaluator is literally *that of complete ignorance within $\boldsymbol{\mathcal{M}}_{\mathrm{co}}$, then the ex-ante preference for x over y seems defensible. Note, however, that for x to be preferred to y by some voter with top θ_i in U, i's preference must be very special; for instance, almost any ellipse with center at θ_i that includes x will also include y. Thus, presumably, for a more finely elicited epistemic state E the upper support $m^+_E(x, y)$ would be less, even much less, than the converse upper support $m^+_E(y, x)$.*

While attractively simple, the usefulness and appropriateness of the complete ignorance assumption is thus challenged by situations as in Example 1. The applicability of the plain convex model is limited because it implies only extremely weak restrictions on the comparison of non-top alternatives. At the same time, examples such as Example 1 show that the distribution of tops can suggest restrictions on preferences over non-top alternatives *in the aggregate*. Heuristically, similar tops tend to give rise to similar comparisons among any pair of non-top alternatives. In the following, we offer two conceptually distinct yet complementary approaches to make this heuristic precise. The first assumes that the distribution of the substitution-complementation structure of preferences is stochastically independent of the distribution of tops; to formalize this, it is assumed that voters' preferences are quadratic. The second is based on the more direct assumption that in estimating the range of 'plausible' majority margins, voters' tops can be bi-partitioned into convex similarity clusters.

3.2 Common Structure: The Quadratic Model

Say that a preference $\succcurlyeq$ on X is *quadratic* if it can be represented by a utility function of the form

$$u_\theta(x) \;=\; (x-\theta)^T \cdot \mathcal{Q} \cdot (x-\theta), \tag{3.2}$$

for some $\theta \in X$ and a negative definite, symmetric $L \times L$ matrix $\mathcal{Q}$. Geometrically, the representation in (3.2) means that the indifference curves are generated from circles with center θ by a common affine transformation; in particular, they are ellipsoids. Quadratic preferences can be viewed as (second-order) Taylor approximations of any underlying smooth preference around the top. In particular, the (globally constant) cross-partial derivatives given by $\mathcal{Q}$ capture the specific pattern of local complementarities and/or substitutabilities between different goods. Denote by $\boldsymbol{\mathcal{M}}_{\text{quad}}$ the model consisting of all profiles of quadratic preferences on X, the *plain quadratic model*. Evidently, for all tops $\theta \in X$ and all $x, y \in X$ such that θ, x, y are not collinear, there exist quadratic preferences $\succcurlyeq, \succcurlyeq'$ both with top θ such that $x \succ y$ and $y \succ' x$. By consequence, we have:

Fact 3.1. *The model $\boldsymbol{\mathcal{M}}_{\text{quad}}$ is a rich submodel of $\boldsymbol{\mathcal{M}}_{\text{co}}$. In particular, the two models induce the same ex-ante majority relation and* $\mathrm{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{quad}}) = \mathrm{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{co}})$.

Thus, also from the perspective of the ex-ante Condorcet solution the plain quadratic model can be viewed as representative of the plain convex model. In particular the 'generic plurality' conundrum posed by Example 1 continues to apply to the plain quadratic model. The great advantage of the quadratic model is that it allows for a clear separation between the preference top and the preference *structure* (described by the quadratic from $\mathcal{Q}_i$.

The tightest way to formalize the idea that voters (with distinct tops) have 'similar' preferences is to require that their preferences be quadratic with the *same* quadratic form, i.e. indifference curves of different voters can be obtained from each other by translation. Specifically, say that a profile $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ of quadratic preferences with tops $(\theta_1, ..., \theta_n)$ is *uniform* if there exists a quadratic form $\mathcal{Q}$ (independent of i) such that, for all i, the preference $\succcurlyeq_i$ is represented by

$$u_{\theta_i}(x) \;=\; (x-\theta_i)^T \cdot \mathcal{Q} \cdot (x-\theta_i).$$

Since a preference is quadratic if and only if is obtained from a Euclidean preference (with circles as indifference curves) by an affine transformation, we have the following result.

Fact 3.2. *Let $x, y \in X$ be any two distinct alternatives, and $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$ a uniform profile of quadratic preferences with tops $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$. Then, there exists a (Euclidean) half-space $H \subseteq \mathbb{R}^L$ such that the hyperplane ∂H passes through the midpoint between x and y, and*

$$\{\theta_i \,|\, x \succ_i y\} \; \subseteq int(H) \quad \text{and} \quad \{\theta_i \,|\, y \succ_i x\} \; \subseteq int(H^c), \tag{3.3}$$

where H^c is the complement of H in $\mathbb{R}^L$. Conversely, for any (Euclidean) half-space H that separates x from y such that ∂H passes through the midpoint between x and y, there exists a uniform profile of quadratic preferences that satisfies (3.3).

Denote by $\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}} \subseteq \boldsymbol{\mathcal{M}}_{\text{quad}}$ the set of all uniform profiles and observe how the uniformity assumption solves the puzzle posed by Example 1 above. By an intermediate value argument, one can choose y so that at least $\frac{n-3}{2}$ tops in U are on either side of the line connecting x and y. Then, any half-space H through the midpoint between x and y that contains y also contains at least $\frac{n-3}{2}$ tops. Thus, $m^-_{(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(y, x) = \frac{n-3}{2}$ while $m^-_{(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(x, y) = 2$. It follows that x cannot be an ex-ante Condorcet winner at this profile in the uniform quadratic model.

While the uniform quadratic model $\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}}$ gives a satisfactory answer in situation as described in Example 1, it is arguably too restrictive in applications. In particular, in any uniform profile, the preference of *one* voter determines the entire preference ordering of all voters given their respective tops. A more permissive condition requires only that, conditional on each top, the distribution of the preference structure of all voters with that top is the same. Specifically, denote by $\mu|_\theta$ the distribution over the quadratic forms of all voters with top θ, and say that a profile of quadratic preferences is *homogeneous* if $\mu|_\theta = \mu|_{\theta'}$ for all θ, θ' that occur as tops in the profile. Denote by $\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}}$ the family of all homogeneous profiles of quadratic preferences, and observe that $\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}} \subseteq \widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}} \subseteq \boldsymbol{\mathcal{M}}_{\text{quad}}$. Observe that the homogeneity assumption in effect requires admissible profiles to have a product structure over tops and quadratic forms; in other words, the conditional distributions of tops conditional on any fixed quadratic form are identical. From this, one easily infers that:

Fact 3.3. *The ex-ante majority relations of the uniform and the homogeneous quadratic model coincide; in particular,* $\text{CW}(\boldsymbol{\theta}, \widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}}) = \text{CW}(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})$.

(Proof in appendix.)

It turns out that the ex-ante Condorcet winners in the uniform and the homogeneous quadratic models coincide and are closely related to the *Tukey median* (Tukey, 1975). For all $x \in X$, denote by $\mathcal{H}_x$ the family of all Euclidean half-spaces that contain x (i.e. the family of all sets of the form $\{y \in X : a \cdot y \geq a \cdot x\}$ for some non-zero vector $a \in \mathbb{R}^L$). For every $x \in X$ and all profiles $\boldsymbol{\theta}$, denote by

$$\mathfrak{d}(x; \boldsymbol{\theta}) := \min_{H \in \mathcal{H}_x} \#(\boldsymbol{\theta} \cap H)$$

the *Tukey depth* of x at the profile $\boldsymbol{\theta}$. Intuitively, the Tukey depth measures the 'centrality' of x with respect to the profile of tops: the larger $\mathfrak{d}(x; \boldsymbol{\theta})$ the more tops θ_i lie in every direction viewed from x, and $\mathfrak{d}(x;\boldsymbol{\theta})=0$ means that x can be separated from the entire set of tops $\boldsymbol{\theta}$ by a hyperplane. Denote by $\mathfrak{d}(\boldsymbol{\theta}) := \max_{x\in X}\mathfrak{d}(x;\boldsymbol{\theta})$ the maximal Tukey depth over X. The *Tukey median rule* selects, for every profile $\boldsymbol{\theta}$, the alternatives that attain this maximal depth:

$$T(\boldsymbol{\theta}) \ := \ \arg\max_{x\in X}\mathfrak{d}(x;\boldsymbol{\theta}) \ = \ \{x\in X \mid \mathfrak{d}(x;\boldsymbol{\theta})=\mathfrak{d}(\boldsymbol{\theta})\}.$$

Our first main result involves the following refinement. For all profiles $\boldsymbol{\theta}$ and all x, denote by $\mathcal{H}^*_x := \{H \ni x : \#(\boldsymbol{\theta}\cap H)=\mathfrak{d}(\boldsymbol{\theta})\}$. A Tukey median $x\in T(\boldsymbol{\theta})$ is *strict* if, for no $y\in T(\boldsymbol{\theta})$, $\mathcal{H}^*_y \subsetneq \mathcal{H}^*_x$. The set of **strict Tukey medians** is denoted by $T^*(\boldsymbol{\theta})$.

Theorem 1. *For all profiles $\boldsymbol{\theta}$,* $\mathrm{CW}(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}})$ *is non-empty; moreover, every element of* $\mathrm{CW}(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}})$ *is a Tukey median, and indeed a strict one. Conversely, every strict Tukey median is an element of* $\mathrm{CW}(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}})$*, i.e.* $\mathrm{CW}(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}})=T^*(\boldsymbol{\theta})$.

(Proof in appendix; see also the remarks on the proof idea in the following subsection.)

3.3 The Similarity Hypothesis

An alternative but related approach to formalize the idea that agents with similar tops will tend to rank non-top alternatives similarly is through a 'similarity hypothesis' directly imposed on the estimated majority intervals (and not indirectly via structural assumptions on the underlying model of possible preference profiles). The present alternative approach can thus be viewed as a 'reduced-form' approach to representing a social evaluators epistemic state.

For a profile to count as a *plausible basis* for estimating the range of majorities for a particular pair of distinct alternatives we require it to satisfy the following condition. Say that a profile of preferences $\succcurlyeq = (\succcurlyeq_1,...,\succcurlyeq_n)$ with tops $(\theta_1,...,\theta_n)$ satisfies the *similarity hypothesis for the pair* $x,y\in X$ of distinct alternatives, if there exists a partition of X into two convex sets (hence half-spaces) H_x and H_y such that

(i) $x\in H_x$ and $y\in H_y$,

(ii) $\#\{i : x\succ_i y\} \geq \#\{i : y\succ_i x\} \implies$

$$\#\{i : x\succ_i y \text{ and } \theta_i\notin H_x\} \leq \#\{i : y\succ_i x \text{ and } \theta_i\notin H_y\}.$$

Intuitively, by condition (i) the two half-spaces represent a linear classification into the 'x-supporters' (the tops of those voters who prefer x to y) and the 'y-supporters;' condition (ii) allows for some misclassification but bounds its extent.

Denote by $\mathrm{SH}_{x,y;\boldsymbol{\theta}}$ the set of ex-post preference profiles satisfying the similarity hypothesis for x and y at the profile $\boldsymbol{\theta}$, and let

$$\widehat{m}^-_{(\boldsymbol{\theta},\mathrm{SH})}(x,y) \ := \ \min\#\{i : x\succ_i y \text{ for some } \succcurlyeq\in\boldsymbol{\mathcal{M}}_{\text{co}}\cap\mathrm{SH}_{x,y;\boldsymbol{\theta}}\}$$

be the 'plausible lower majority' of x against y. Accordingly, we refer to

$$xR_{(\boldsymbol{\theta},\mathrm{SH})}y \ :\Longleftrightarrow \ \widehat{m}^-_{(\boldsymbol{\theta},\mathrm{SH})}(x,y) \geq \widehat{m}^-_{(\boldsymbol{\theta},\mathrm{SH})}(y,x), \tag{3.4}$$

as the **plausible ex-ante majority relation** and denote by $\mathrm{CW}(\boldsymbol{\theta},\mathrm{SH})$ its maximal in X.

Theorem 2. *For all profiles $\boldsymbol{\theta}$, $\mathrm{CW}(\boldsymbol{\theta}, \mathrm{SH})$ is non-empty and $x \in \mathrm{CW}(\boldsymbol{\theta}, \mathrm{SH})$ if and only if x is a strict Tukey median.*

Remark 1. One might consider other, prima-facie more restrictive versions versions of the similarity hypothesis by requiring equality of the extent of misclassification in condition (ii), or even rule out any misclassification at all. This would not change the plausible majority margins, i.e. the $\widehat{m}^{-}_{(\boldsymbol{\theta},\mathrm{SH})}(x, y)$; hence, it would also neither change the ex-ante majority relation nor the ex-ante Condorcet winners.

Remark 2. One might also consider strengthening condition (i) by requiring symmetry of the classification, i.e. by requiring that the midpoint between x and y be contained both in H_x and H_y. This in fact changes the plausible majority intervals and the ex-ante majority relation, but it does not change the ex-ante Condorcet winners (as shown in Lemma B.8 in the appendix). Indeed, the majority margins now coincide with those of the uniform and homogeneous quadratic model by Fact 3.2.

The proof of Theorem 2 (provided in the appendix) proceeds in three main steps. First, it is observed that the ex-ante majority relation $R_{(\boldsymbol{\theta},\mathrm{SH})}$ can be characterized in terms of the *relative depth* of two alternatives vis-á-vis each other, i.e.,

$$xR_{(\boldsymbol{\theta},\mathrm{SH})}y \iff \min_{H \in \mathcal{H}_x,\, y \notin H} \#(\boldsymbol{\theta} \cap H) \geq \min_{H \in \mathcal{H}_y,\, x \notin H} \#(\boldsymbol{\theta} \cap H). \tag{3.5}$$

Using this one can show that, for all profiles $\boldsymbol{\theta}$, the relation $R_{(\boldsymbol{\theta},\mathrm{SH})}$ is quasi-transitive (i.e. its strict part is transitive), and that its maxima coincide with the strict Tukey median. Finally, the set of maxima is shown to be non-empty by an argument based on Zorn's lemma. (We cannot invoke standard arguments here because the upper contour sets are not generally open.) The full proof requires a number of intermediate steps that are detailed in the appendix.

Theorem 1 follows from Theorem 2 by the fact that the ex-ante majority relation of the homogeneous quadratic model coincides *locally* with the plausible ex-ante majority relation defined by (3.4); this implies that the ex-ante Condorcet winners of the homogeneous quadratic model coincide with $\mathrm{CW}(\boldsymbol{\theta}, \mathrm{SH})$ (see Lemma B.8 in the appendix).

Example 2. *For illustration, consider the following example. Suppose that there are five voters with tops θ_i, $i = 1, ..., 5$, respectively, that form a pentagon as shown in Figure 2. The ex-ante Condorcet winners are given by the points in the inner convex pentagon marked in red.*[9] *Fig. 2 also shows a point y and its associated upper contour set with respect to the plausible ex-ante majority relation (in blue). Note in particular that the points x and y have the same (absolute) Tukey depth but different relative depth with $xP_{(\boldsymbol{\theta},\mathrm{SH})}y$.*
If we compare x and y in Fig. 2 with respect to the ex-ante majority relation of the homogeneous quadratic model, we again obtain $xP_{(\boldsymbol{\theta},\mathrm{SH})}y$; but if we look at alternative x', say, we get both $x'R_{(\boldsymbol{\theta},\mathrm{SH})}y$ and $yR_{(\boldsymbol{\theta},\mathrm{SH})}x'$ but $x'P_{(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})}y$ due to Fact 3.2. This shows that the ex-ante

[9] This can be verified from the following observations. First, any line passing through the inner red pentagon has at least two tops on either side; on the other hand, for any point outside the inner pentagon there is a Euclidean half-space containing that point and at most one top. In particular, the maximal Tukey depth is $\mathfrak{d}(\boldsymbol{\theta}) = 2$; all points in the convex hull of the tops that are not in this inner pentagon have depth one; and all points outside the convex hull of the tops have depth zero.

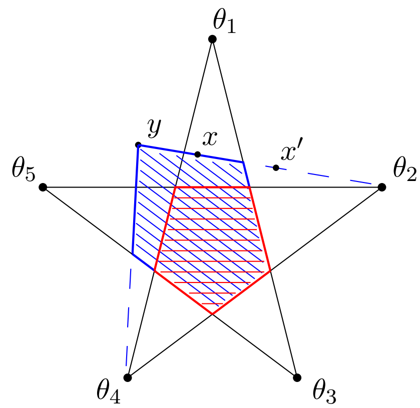

Figure 2: The pentagon

majority relation of the homogeneous quadratic model generally differs from the plausible ex-ante majority relation (even though the corresponding ex-ante Condorcet winners coincide by Theorems 1 and 2).

While every Tukey median is strict in Example 2, it is an open question if this is the case generally. It must be the case whenever Tukey medians are unique (because strict Tukey medians always exist). Demange (1982) has in fact shown such uniqueness whenever voters' tops are continuously distributed with a convex support.

3.4 Why the Tukey Median?

The Tukey median is a standard multi-dimensional median in multi-variate statistics. But there are also other multi-dimensional medians, in particular those based on L_p-norms. Prima facie, these appear natural also in the current setting. Specifically, for $1 \leq p < \infty$, let

$$||x||_p \;:=\; \left(\sum_{\ell=1}^{L} |x^\ell|^p \right)^{1/p}$$

denote the L_p-norm, and consider, for all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, the 'scoring functions' $\Delta_p(\cdot, \boldsymbol{\theta})$ defined by $\Delta_p(x; \boldsymbol{\theta}) \;:=\; \sum_{i=1}^{n} ||x - \theta_i||_p$. The corresponding solution to the budget allocation problem on X is given by minimizing this scoring function, i.e.,

$$C_{p-\mathrm{med}}(\boldsymbol{\theta}) \;:=\; \arg\min_{x \in X} \Delta_p(x; \boldsymbol{\theta}).$$

Because in the one-dimensional case, $C_{p-\mathrm{med}}$ coincides for all p with the standard median, we will refer to the corresponding solution as the *p-median*.

The p-medians are majority admissible in either the plain convex model and the homogeneous quadratic model, for all $p \in [1, \infty)$. They are attractive from a social choice perspective also because they represent a social evaluation by means of imputed utility functions which can be viewed as representing an ex-ante expected utility of a voter with known top (namely, the L_p-distance to the top).

But while the p-medians adequately capture the knowledge of convexity of preferences they arguably fail to fully capture the ignorance about the shape of individual preferences beyond that. Arguably, a solution that fully reflects that ignorance should satisfy the following condition of 'qualified' affine equivariance. Say that a *solution* $C(\boldsymbol{\theta}) \subseteq X$ satisfies **qualified affine equivariance** if, for all affine mappings $\lambda : \mathbb{R}^L \longrightarrow \mathbb{R}^L$ with full rank, and for all profiles $\boldsymbol{\theta} \in X^n$ such that $\lambda(\boldsymbol{\theta}) \in X^n$,

$$[C(\boldsymbol{\theta}) \subseteq co(\boldsymbol{\theta}) \ \& \ C(\lambda(\boldsymbol{\theta})) \subseteq co(\lambda(\boldsymbol{\theta}))] \implies C(\lambda(\boldsymbol{\theta})) = \lambda(C(\boldsymbol{\theta})). \tag{3.6}$$

Indeed, the information about ex-post preference profiles restricted to the convex hulls in the convex model is equivariant under the affine transformation; hence, the choice should also be equivariant *provided* it is restricted to the convex hull of the tops.[10]

While the p-medians violate the qualified affine equivariance condition (3.6) for all $p \in [1, \infty)$, the strict and non-strict Tukey medians as well as plurality rule satisfy it. Another simple rule that satisfies (qualified) affine equivariance is the *mean rule* which selects the coordinate-wise average of the voter's top allocations.[11]

Fact 3.4. *The strict and non-strict Tukey median rules as well as the mean rand plurality rules satisfy the qualified affine equivariance condition (3.6).*

(Proof in appendix.)

The following general 'impossibility' result shows that a price has to be paid if one insists on majority admissibility and the qualified affine equivariance condition as axiomatic requirements in the convex model.

Proposition 5. *There is no upper hemicontinuous solution that always chooses from the convex hull of the tops and satisfies both the qualified affine equivariance condition (3.6) and majority admissibility either with respect the plain convex model or, a fortiori by (2.3), with respect to the homogeneous quadratic model.*

(Proof in appendix.)

Proposition 5 has the corollary that no majority admissible and affinely equivariant solution can admit a utilitarian representation with continuous and concave imputed utility functions, since such a solution would automatically be upper hemicontinuous.[12] This has the further consequence that a viable solution such as the (strict) Tukey median cannot be expected to satisfy the following standard reinforcement property. Let $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ be two profiles

[10]Outside the convex hull, the mapping λ may not map feasible alternatives to feasible alternatives; so in this sense, the information about preferences outside the respective convex hull does not match. Restricting attention to the convex hulls thus appeals to a weak IIA-style argument.

[11]The mean rule corresponds to the minimization of the scoring function $\sum_{i=1}^{n} (||x - \theta_i||_p)^q$ with $p = q = 2$.

[12]Plurality rule has a utilitarian representation with the non-continuous imputed utility functions that assign unit utility to the top allocation and zero to all other allocations, resepctively.

of tops corresponding to two disjoint sets of voters, and denote by $\boldsymbol{\theta} \sqcup \boldsymbol{\theta}'$ the profile of the combined electorate; a solution $C(\cdot)$ satisfies **reinforcement** if

$$C(\boldsymbol{\theta}) \cap C(\boldsymbol{\theta}') \neq \emptyset \implies C(\boldsymbol{\theta} \sqcup \boldsymbol{\theta}') = C(\boldsymbol{\theta}) \cap C(\boldsymbol{\theta}'). \tag{3.7}$$

Both the strict and non-strict Tukey medians violate (3.7) as shown by the following example.

Example 3. *Let $L = 3$, $Q = 4$, and consider the following two profiles each giving the tops of two distinct sets of 10 voters (cf. Fig. 3): the profile $\boldsymbol{\theta}$ (marked in red in Fig. 3) contains four voters with top $(2,2,0)$, three voters with top $(2,0,2)$, and three voters with top $(4,0,0)$; the profile $\boldsymbol{\theta}'$ (marked in blue in Fig. 3) contains four voters with top $(2,2,0)$, three voters with top $(2,0,2)$, and three voters with top $(0,2,2)$. As is easily verified, we have $T(\boldsymbol{\theta}) = T(\boldsymbol{\theta}') = \{(2,2,0)\}$. Reinforcement would thus require the choice of $(2,2,0)$ also at the combined profile $\boldsymbol{\theta} \sqcup \boldsymbol{\theta}'$, but in fact we have $T(\boldsymbol{\theta} \sqcup \boldsymbol{\theta}') = \{(2,1,1)\}$ with the point $(2,1,1)$ achieving a Tukey depth of $9/20$. Note that the strict and non-strict Tukey median coincide due to their uniqueness at the relevant profiles in this example.*

From a traditional social choice perspective, the violation of reinforcement might be viewed as a serious normative drawback of the (strict) Tukey median. However, we submit that this conclusion is inappropriate in the present, epistemic setting. For here, the similarity hypothesis implies an epistemic interaction between the information about the profiles in the two subpopulations under consideration. This interactions explains naturally the potential failure of reinforcement. For instance, consider in Example 3 the comparison of the allocations $(2,1,1)$ vs. $(2,2,0)$. By the mere convexity of ex-post preferences the support count in the combined profile is 6 voters for $(2,1,1)$ against 8 voters for $(2,2,0)$; but due to the similarity hypothesis the point $(2,1,1)$ receives the additional support either of the three voters at $(4,0,0)$ or the three voters at $(0,2,2)$ (see Figure 3).

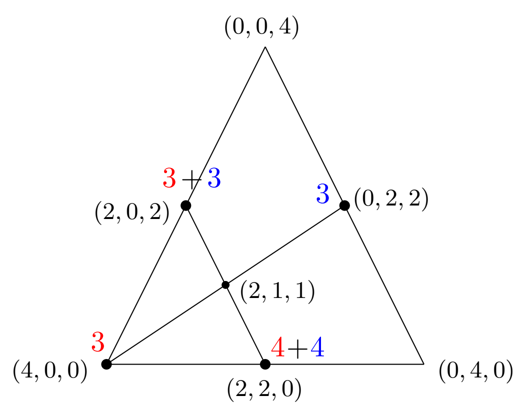

Figure 3: The (strict) Tukey median violates reinforcement

To compare our proposal of the Tukey median solution to the standard 'spatial' voting approach, consider the Euclidean model $\boldsymbol{\mathcal{M}}_{\text{Euclid}}$ of all profiles of preferences representable by the negative L_2-distance to the respective tops; this amounts to the spatial voting model adapted to the present resource allocation problem with a convex feasibility constraint (typically, the standard literature considers an unconstrained model). Majority admissibility in the Euclidean model is still rather weak; in particular, in view of the classical result by McKelvey (1979), it is generically multi-valued. Moreover, in that model the appropriate invariance requirement is one of (qualified) invariance to isometries rather than affine transformations. This is in fact compatible with the reinforcement condition. Among the p-medians, these three conditions characterize the 2-median which minimizes the sum of Euclidean distances to the individual tops. Thus, the natural candidate for a frugal optimum in the Euclidean model is the 2-median, not the Tukey median.

The frugal perspective thus contrasts starkly with the standard approach of the literature which uses the Euclidean model to apply different voting rules and solution concepts to profiles of complete preferences. In particular, it is straightforward to observe that the minimax voting rule is equivalent to the choice of the Tukey median of voters' tops, see in particular Kramer (1977). Other important contributions to this literature are Demange (1982) and Caplin and Nalebuff (1988) which analyze mathematical properties of the Tukey median. For a recent application to the 'political economy' of the firm, see Crès and Tvede (2021).

4 The Separably Convex Model

Consider the situation depicted in Figure 4. Assume that in the type profile $(\theta_1; p_1, \theta_2; p_2, \theta_3; p_3)$ none of tops receives a majority while θ_1 is the plurality winner, i.e. $p_i < 1/2$ for $i = 1, 2, 3$ and $p_1 > p_2, p_3$; moreover, suppose that the three tops are not collinear, but θ_3 is 'nearly' on the line segment between θ_1 and θ_2. Then, the strict Tukey median chooses the plurality winner θ_1. The rationale for choosing θ_1 over θ_3 is based on a possible preference of θ_1 over θ_3 by the voters with top θ_2. But if θ_3 is nearly on the line segment between θ_1 and θ_2 this is a tight affair and can hold only for very special convex preferences.

Thus, the choice of θ_1 is a risky implication of the regularized convex model. And indeed, it may be demonstratively mistaken if more is known about individual preferences, for instance, if in addition to convexity it is known that individual ex-post preferences are separable. Specifically, in the present section we will study the 'separably convex' model and show that under this model the ex-ante Condorcet winner at the type profile shown in Fig. 4 is θ_3, as intuition suggests.

For expository convenience and generality, we will discretize the space of feasible allocation, i.e. consider the space

$$X := \left\{ x \in \mathbb{Z}^L \mid \sum_{\ell=1}^{L} x^\ell = Q \text{ and } x^\ell \in [q_-^\ell, q_+^\ell] \text{ for all } \ell = 1, ..., L \right\}, \tag{4.1}$$

where $\mathbb{Z}$ is the set of all integers; the adaption of the following analysis to the continuous case is given in Appendix A.

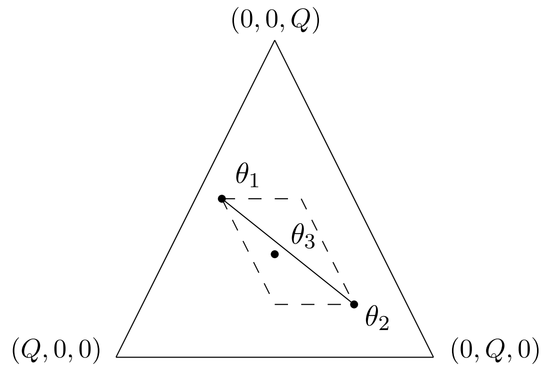

Figure 4: Almost collinearity

4.1 Definition

Intuitively, separable convexity states that if an individual would disapprove a unit transfer of expenditure from good j to good k at an allocation x, then this individual would also disapprove such transfer at any allocation that has less expenditure on j and more expenditure on k; formally:

Definition (Separable Convexity) *For any allocation $x \in X$ denote by $x_{(kj)}$ the allocation that results from x by transferring one unit of money from good j to good k, i.e. $x^k_{(kj)} = x^k + 1$, $x^j_{(kj)} = x^j - 1$ and $x^\ell_{(kj)} = x^\ell$ for all $\ell \neq k, j$. Say that a preference order $\succcurlyeq$ on X is* separably convex *if $x \succ x_{(kj)}$ implies $y \succ y_{(kj)}$ for all k, j, x, y such that $y^k \geq x^k$ and $y^j \leq x^j$.*

Separable convexity contains two special cases: (i) 'linear' convexity (i.e. single-peakedness) and (ii) separability. Case (i) is given by the additional condition that $x^\ell = y^\ell$ for all $\ell \neq k, j$ (see the left panel in Figure 5), while case (ii) is given by the additional condition that $y^k = x^k$ and $y^j = x^j$. Separable convexity integrates these two requirements but is somewhat stronger than the logical conjunction of linear convexity and separability. To see this, note that separability is vacuous for $L = 3$ due to the budget constraint on the domain of feasible allocations X over which $\succcurlyeq$ is defined. The right panel in Fig. 5 shows the general case for $L = 3$, $k = 3$ and $j = 2$, combining convexity and separability.

Denote by $\mathcal{R}_{\text{sepco}}$ the set of all separably convex weak orders on X with a unique top. The leading example of such weak orders are preferences with an additively separable utility representation of the form

$$u(x) \;=\; u(x^1, ..., x^L) \;= \sum_{\ell=1}^{L} u^\ell(x^\ell), \tag{4.2}$$

where the $u^\ell : \mathbb{R} \to \mathbb{R}$ are strictly increasing and concave for all $\ell = 1, ..., L$. For future reference, we denote the set of all preferences with such an additively separable and concave representation by $\mathcal{R}_{\text{addco}}$.

The main object of analysis of the present section is the **(plain) separably convex model $\boldsymbol{\mathcal{M}}_{\text{sepco}}$** $:= \cup_{n\in\mathbb{N}}(\mathcal{R}_{\text{sepco}})^n$. (We omit the term 'plain' in the following whenever no

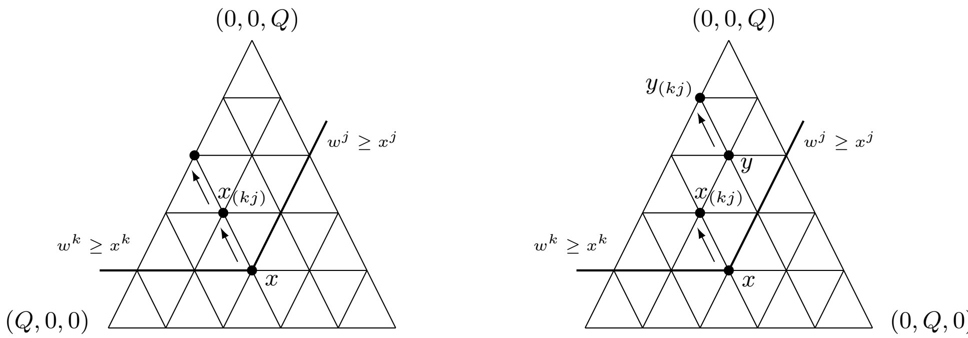

Figure 5: Separable convexity

confusion can arise.) Before proceeding, we note that the (plain) additively separable model $\boldsymbol{\mathcal{M}}_{\text{addco}} := \cup_{n\in\mathbb{N}}(\mathcal{R}_{\text{addco}})^n$ is rich submodel of the separably convex model, i.e. it induces exactly the same intervals of possible support counts (see Lemma B.2 in the appendix).

4.2 Inferred Preferences

First, we study the information about the unobservable preferences that can be inferred from an individual top under the background assumption of separable convexity. For instance, if a voter submits the top alternative x in the right panel of Fig. 5, by virtue of the background assumption of separable convexity the social evaluator can infer that this voter prefers y to y_{kj}. In general, for all $\theta \in X$, and all distinct $x, y \in X$, let

$$x >_{\theta}^{\text{sepco}} y \;:\Longleftrightarrow\; x \succ y \;\text{ for all } \succcurlyeq\, \in \mathcal{R}_{\text{sepco}} \text{ with top } \theta. \tag{4.3}$$

We will refer to $>_{\theta}^{\text{sepco}}$ defined by (4.3) as the voter's **inferred preference relation** (under the separably convex model); observe that it is a *partial* order. For all $x, y \in \mathbb{Z}^L$, let

$$[x, y] \;:=\; \left\{ w \in \mathbb{Z}^L \mid \text{for all } \ell = 1, ..., L,\; x^\ell \leq w^\ell \leq y^\ell \text{ or } y^\ell \leq w^\ell \leq x^\ell \right\}.$$

We will refer to $[x, y]$ as the *box spanned by x and y* and to the elements of $[x, y]$ as the (not necessarily feasible) allocations *between x and y*. Moreover, say that two allocations $x, y \in X$ are *neighbors* if and only if $[x, y] \cap X = \{x, y\}$. Denote by Γ_{res} the graph that results from connecting all neighbors in X with an edge. (Observe that Fig. 5 above depicts exactly this graph.) Importantly, the betweenness can be derived from the graph in that

$$w \in [x, y] \iff w \text{ is on a shortest } \Gamma_{\text{res}}\text{-path connecting } x \text{ and } y. \tag{4.4}$$

The metric in the graph Γ_{res} will also play a significant role. In resource terms, it is easily seen that

$$d(x, y) := \frac{1}{2} \sum_{\ell=1}^{L} |x^\ell - y^\ell|;$$

in other words, the graph distance of Γ_{res} coincides with the natural (normalized) 'resource' metric on X, i.e. the L_1-metric. In terms of the graph distance, property (4.4) can thus be re-written as

$$[x,y] \;=\; \{w \in X \mid d(x,y) = d(x,w) + d(w,y)\} \tag{4.5}$$

for all $x, y \in X$. The following result is fundamental to the subsequent characterization of the ex-ante Condorcet winner.

Lemma 4.1. *For all $\theta \in X$ and all distinct $x, y \in X$,*

$$x >_{\theta}^{\text{sepco}} y \iff x \in [\theta, y]. \tag{4.6}$$

Proof. We show that x is preferred to y by any separably convex preference order $\succcurlyeq$ with top θ whenever x is between θ and y. Thus, consider a shortest Γ_{res}-path between θ and y through x. Let w and $w_{(kj)}$ be any two neighbors on that path such that $w \in [x,y]$ and $w_{(kj)} \in [w,y]$ as in Figure 6. By construction, we have $y^k \geq \theta^k$ and $y^j \leq \theta^j$. If θ is the top alternative of $\succcurlyeq$, we clearly have $\theta \succ \theta_{(kj)}$; hence by separable convexity, $w \succ w_{(kj)}$ (see Fig. 6). Since the argument applies to all neighbors w and $w_{(kj)}$ on the chosen shortest path between x and y, we obtain $x \succ y$ by transitivity.

This proves sufficiency of the betweenness property on the right hand side of (4.6) for an inferred strict preference of x over y; necessity is demonstrated by the explicit construction of a suitable separably convex preference order in the appendix.

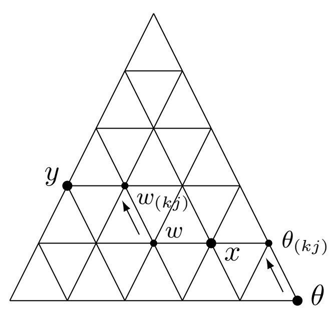

Figure 6: Betweenness implies inferred preference

□

Consider again our initial example in Fig. 4 above. By Lemma 4.1, we obtain that all voters of type 1 (those with top θ_1) can be inferred to prefer θ_3 to θ_2 under separable convexity; in fact for these voters, any point in the dashed area is preferred to θ_2. Similarly, all voters of type 2 (those with top θ_2) can be inferred to prefer θ_3 (and any point in the dashed area) to θ_1. Consequently, θ_3 is the only majority admissible alternative among the three tops in the profile $(\theta_1, \theta_2, \theta_3)$ under separable convexity (and indeed the unique solution under the separably convex model, as we shall see below).

4.3 Majority Admissibility in the Separably Convex Model

Majority admissibility has substantially stronger implications in the separably convex model than in the convex models; nevertheless, it may still not be decisive if $L > 2$. We illustrate this here in special cases (for further examples, see Appendix A.2). Call a subset $D \subseteq X$ an *ordered domain* (with respect to the separably convex model) if, for all three elements from D, one is between the other two. The terminology is justified by the fact an ordered domain D can be ordered by a linear order $\triangleright_D$ such that $x \triangleright_D y \triangleright_D z$ if and only if $y \in [x, z]$. Furthermore, say that D is *line-like* if, for some fixed pair of coordinates $j, k \in \{1, ..., L\}$, any two elements of D agree in all coordinates $\ell \notin \{j, k\}$. Evidently, every line-like domain is also ordered, but the converse is not true. We have the following result.

Proposition 6. *Let $D \subseteq X$ be an ordered domain.*

(i) *Suppose that $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is such that* supp $\boldsymbol{\theta} \subseteq D$*, then*

$$\mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \cap D \ \subseteq \ C_{1-\mathrm{med}}(\boldsymbol{\theta})$$

with equality if n is odd.

(ii) *If D contains three elements that are not collinear, there exists $\boldsymbol{\theta}$ with* supp $\boldsymbol{\theta} \subseteq D$ *such that*

$$C_{p-\mathrm{med}}(\boldsymbol{\theta}) \ \subseteq D \setminus C_{1-\mathrm{med}}(\boldsymbol{\theta})$$

for all $p > 1$.

(iii) *Let $D \subseteq X$ be line-like domain and $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ a profile with* supp $\boldsymbol{\theta} \subseteq D$ *and n odd, then*

$$\mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \ = \ C_{p-\mathrm{med}}(\boldsymbol{\theta}) \ = \ \{\theta_{\mathrm{med}}\}$$

for all $p \geq 1$, where θ_{med} is the (unique) median element of $\boldsymbol{\theta}$ with respect to the order $\triangleright_D$.

(Proof in appendix.)

By the third part of Proposition 6 and Proposition 1, if an ex-post Condorcet winner exists for a line-like profile with an odd number of voters, it must be the unique median top (however, even for line-like profiles an ex-post Condorcet winner need not exist, as shown by Example 7 in Appendix A.3). The first part of Proposition 6 asserts that the majority admissible elements of an ordered domain coincide with the 1-median; the second part shows that, by contrast, all p-medians with $p > 1$ sometimes choose only alternatives that are not majority admissible under the separably convex model. The general conclusion from this is that, among all p-medians, only the 1-median remains a viable candidate for the separably convex model.

For illustration, consider Figure 7 which depicts an ordered domain D. Suppose that three types of agents have their top within D, specifically at θ_1, θ_2 and θ_3, respectively. Suppose that no type represents an absolute majority. Then, the 1-median of the corresponding profile, and the only majority admissible point within D, is θ_2. However, outside D there might exist other majority admissible alternatives, for instance the point w; therefore, the first inclusion in part (i) of Proposition 6 is restricted to the majority admissible elements *within* the set D. Part (ii) is shown by considering profiles with support $\{\theta_1, \theta_2, \theta_3\}$ as in Fig. 7 but with non-uniform mass; specifically, if θ_1 has the uniquely largest mass among them but less than 50%, all p-medians uniquely select the 'plurality winner' θ_1. But clearly, θ_1 is not majority admissible at such profile because more than 50% of the voters can be inferred to strictly prefer θ_2 in the separably convex model.

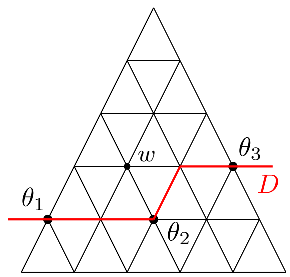

Figure 7: Majority admissibility in an ordered domain

4.4 The 'Condorcet Solution:' Local Ex-Ante Condorcet Winners

Under the separably convex model the information contained in the inferred partial order can be recovered from the local comparisons of all neighbors in the graph Γ_{res}, as follows.

Fact 4.1. *For all tops $\theta \in X$, the inferred preference relation $>_\theta^{\text{sepco}}$ coincides with the transitive closure of its restriction to the graph Γ_{res}.*

(Proof in appendix.)

This implies that the necessary majority criterion itself can be recovered from local information:

Fact 4.2. *For all profiles $\boldsymbol{\theta}$, the necessary majority relation $P^{\text{nec}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}$ coincides with the transitive closure of its restriction to the graph Γ_{res}. In particular, if $yP^{\text{nec}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}x$, then there exists a neighbor y' of x such that $y'P^{\text{nec}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}x$.*

(Proof in appendix.)

Fact 4.2 suggests to focus on the restriction of the ex-ante majority relation to local comparisons, as follows. For all profiles $\boldsymbol{\theta}$ and all distinct $x, y \in X$, the **local majority relation** is given by

$$xR^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}y \;:\Longleftrightarrow\; \left[xR_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}y \text{ and } x\Gamma_{\text{res}}y\right] \tag{4.7}$$

with asymmetric part $P^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}$. For all profiles $\boldsymbol{\theta}$, define the set of local ex-ante Condorcet winners by

$$\text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) \;:=\; \{x \in X \mid \text{ for no } y \in X,\; yP^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}x\}.$$

Thus, an alternative is a local ex-ante Condorcet winner if it is not beaten by any neighbor in terms of the ex-ante majority relation. Henceforth, we will refer to the set of local ex-ante Condorcet winners also simply as the **Condorcet solution** (in the separably convex model). Note that, by Fact 4.2, we have

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;\subseteq\; \mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}). \tag{4.8}$$

for all profiles $\boldsymbol{\theta}$. For illustration, consider the alternative w in Fig. 7. While majority admissible, w is beaten by its neighbor θ_2 in terms of ex-ante majority; indeed, the coordinate-wise median θ_2 is the unique Condorcet solution in this example.

4.5 The Condorcet Solution Coincides with the 1-Median

We are ready to state the first main result of this section. A subset $Y \subseteq X$ is called *box convex* if it contains with any two elements the entire interval spanned by them, i.e. $x, y \in Y$ implies $[x, y] \subseteq Y$.

Theorem 3. *For all profiles $\boldsymbol{\theta}$,*

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;=\; C_{1-\mathrm{med}}(\boldsymbol{\theta}).$$

In particular, the set of local ex-ante Condorcet winners in the separably convex model is non-empty and box convex.

We provide the sketch of the proof, the full argument is given in the appendix. First, observe that using Lemma 4.1 we obtain the following tri-partition of X for any pair of neighbors.

Lemma 4.2. *For every pair of neighbors $x, y \in X$ and every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$,*

$$(x \not>^{\mathrm{sepco}}_{\theta_i} y \text{ and } y \not>^{\mathrm{sepco}}_{\theta_i} x) \iff d(x, \theta_i) = d(y, \theta_i). \tag{4.9}$$

In particular, every pair of neighbors induces the following tri-partition of X (see Fig. 8):

$$\begin{aligned}
\{\theta_i : x >^{\mathrm{sepco}}_{\theta_i} y\} &= \{\theta_i : (d(x, \theta_i) = d(y, \theta_i) + 1\}, \\
\{\theta_i : x \not>^{\mathrm{sepco}}_{\theta_i} y \text{ and } y \not>^{\mathrm{sepco}}_{\theta_i} x\} &= \{\theta_i : (d(x, \theta_i) = d(y, \theta_i)\}, \\
\{\theta_i : y >^{\mathrm{sepco}}_{\theta_i} x\} &= \{\theta_i : (d(y, \theta_i) = d(x, \theta_i) + 1\}.
\end{aligned}$$

Using Lemma 4.2, it follows easily that, for all neighbors x, y,

$$x P^{\mathrm{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})} y \iff \Delta_1(x; \boldsymbol{\theta}) \leq \Delta_1(y; \boldsymbol{\theta}).$$

This implies acyclicity of $P^{\mathrm{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})}$, hence non-emptiness of $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ and the agreement of $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ with the *local* minimizers of the aggregate distance. We then observe that, for each top θ, the distance function $d(x, \theta)$ is a separable and convex (in each coordinate) function of x. Hence, aggregate distance $\Delta_1(\cdot; \boldsymbol{\theta})$ is likewise separable and convex

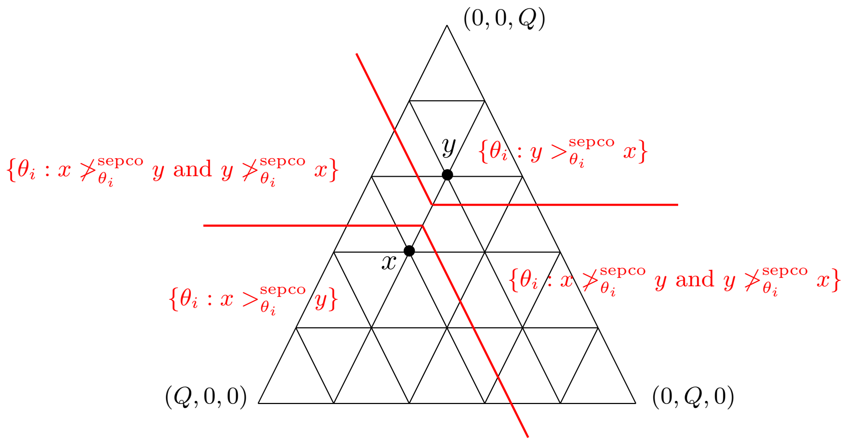

Figure 8: The tri-partition of X for neighbors x and y

in each coordinate. It follows that every local minimum of aggregate distance is a global minimum, and that the set of minimizers is box-convex.

The 'scoring rule' representation of the Condorcet solution as the 1-median provided by Theorem 3 has significant consequences. Most importantly, it reconciles the Condorcet and Borda perspectives in the frugal model under separable convexity; indeed, the Borda perspective can be fleshed out in a natural manner by interpreting distance as negative preference rank, see Appendix A.4. This has further attractive implications. First, it is immediate from Theorem 3 that the ex-ante Condorcet set satisfies the reinforcement property (3.7). Perhaps more surprisingly, it also implies the absence of the no-show paradox, i.e. it can never be harmful for a voter to participate and submit her true top, see Appendix A.6.

In contrast to their local counterpart, *global* ex-ante Condorcet winners need not exist in the separably convex model, see Example 8 in Appendix A.7. We now turn to an application that further supports the local perspective.

4.6 Special Case: The Committee Selection Problem

Consider the following committee selection problem. There are L potential candidates among which a committee of size Q has to be formed where Q is any integer between 1 and L. Formally, the agenda X is thus given by (4.1) with the additional restriction that, for all $\ell = 1, ..., L$, $q^\ell_- = 0$ and $q^\ell_+ = 1$, i.e. that each potential candidate ℓ is either a member of the selected committee, or not. Evidently, 'linear' convexity of preferences plays no role here, and separable convexity reduces to mere *(ordinal) separability* in the sense that the preference for an exchange of a committee member by a non-member is independent of who the other members of the committee are. Formally, let X be the set of all committees with exactly Q members. For all $K, K' \subseteq L$ with $\#K = \#K' = Q - 1$ and all $k, j \in L$ such that $K, K' \subseteq L \setminus \{k, j\}$,

$$K \cup \{k\} \succ K \cup \{j\} \iff K' \cup \{k\} \succ K' \cup \{j\}. \tag{4.10}$$

Thus, candidates can be compared independently of the presence or absence of other candidates. In particular, (4.10) excludes complementaries and team effects resulting from different competencies. This condition has already been considered in the context of committee selection problems, see Lang and Xia (2016) for an overview of the literature. Note that local preferences extract all available information under separability. Accounting for the social evaluator's ignorance, a voter i is treated as indifferent between two adjacent committees $K \cup \{k\}$ and $K \cup \{j\}$ with $K \subseteq L \setminus \{k, j\}$ whenever i's top committee either contains both k and j, or none of them. This seems eminently sensible – with the given background information, how could the selection of candidates different from both k and j throw any light on the selection between these two candidates? The ex-ante majority rule aggregates preferences in a canonical way and picks the committee(s) consisting of Q members which receive the highest popular support, as follows. For each candidate $\ell = 1, ..., L$ and all profiles $\boldsymbol{\theta}$ of top committees, let $n^\ell(\boldsymbol{\theta})$ be the number voters i such that ℓ belongs to i's top committee θ_i ('candidate ℓ's support').

Proposition 7. *A set $K^* \subseteq L$ with Q members constitutes a local ex-ante Condorcet winning committee at the profile $\boldsymbol{\theta}$ if and only if no candidate outside K^* receives more support than some candidate in K^*, i.e. if for all $\ell \notin K^*$*

$$n^\ell(\boldsymbol{\theta}) \;\leq\; \min_{j \in K^*} n^j(\boldsymbol{\theta}).$$

(Proof in appendix.)

Thus, the Condorcet solution in the committee selection problem yields the canonical solution which has been studied in the literature under the name of Q–approval voting (sometimes also more informally referred to as 'bloc' voting). Proposition 7 adds a novel epistemic foundation of Q–approval voting within the frugal aggregation approach.

4.7 Properties of the Condorcet Solution

In this subsection, we provide a simple and powerful characterization of the Condorcet solution that allows one to compute it efficiently.[13] We also derive a number of its basic properties. In the following fix a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ with n voters and denote, for every $\ell = 1, ..., L$ and every $k = 1, ..., n$, by $\theta^\ell_{[k]} \in X$ the k-th smallest vote in coordinate ℓ, that is, the vector $(\theta^\ell_{[1]}, \theta^\ell_{[2]}, ..., \theta^\ell_{[n]})$ results from the values $\theta^\ell_1, \theta^\ell_2, ..., \theta^\ell_n$ simply by re-arranging the latter in ascending order so that $\theta^\ell_{[1]} \leq \theta^\ell_{[2]} \leq ... \leq \theta^\ell_{[n]}$ (possibly with some equalities). Denote by $Q_{[k]} := \sum_{\ell=1}^{L} \theta^\ell_{[k]}$, and let $k^*(\boldsymbol{\theta})$ be the largest $k = 1, ..., n$ such that $Q_{[k]} \leq Q$. Finally, say that the profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *unanimous* if $\theta_1 = \theta_2 = ... = \theta_n$. Note that for a unanimous profile one has $k^*(\boldsymbol{\theta}) = n$ since, evidently, $\theta^\ell_{[1]} = \theta^\ell_{[2]} = ... = \theta^\ell_{[n]} = \theta^\ell_i$ for all $i = 1, ..., n$ and all $\ell = 1, ..., L$. Also observe that $k^*(\boldsymbol{\theta}) < n$ for all non-unanimous profiles.

Theorem 4. *For every non-unanimous profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$, $x \in \mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ if and only if, for all $\ell = 1, ..., L$,*

$$\theta^\ell_{[k^*(\boldsymbol{\theta})]} \leq x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}. \tag{4.11}$$

[13] The result has been obtained independently and in a different context by Freeman *et al.* (2021, Lemma 6.3).

(Proof in appendix.)

Condition (4.11) means that $q^*(\boldsymbol{\theta}) := k^*(\boldsymbol{\theta})/n$ is the *'endogenous' (i.e. profile-dependent) quota* of voters who can be satisfied in all coordinates (of course, these have to be different sets of voters in different coordinates). The following example illustrates this.

Example 4. *Consider the case $L = 3$, $Q = 10$, and a profile $\boldsymbol{\theta}$ with four voters such that $\theta_1 = (5,0,5)$, $\theta_2 = (0,2,8)$, $\theta_3 = (2,6,2)$ and $\theta_4 = (4,3,3)$, say. For the corresponding matrices (θ_i^ℓ) and $(\theta_{[k]}^\ell \,|\, Q_{[k]})$ with $\ell = 1,...,L$ and $i,k = 1,...,n$ we thus obtain*

$$(\theta_i^\ell) \;=\; \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 4 & 3 & 3 \end{pmatrix} \quad \textit{and} \quad (\theta_{[k]}^\ell \,|\, Q_{[k]}) \;=\; \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 3 & | & 7 \\ 4 & 3 & 5 & | & 12 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Since $Q_{[2]} = 7 < 10\ (= Q) < 12 = Q_{[3]}$, we obtain $k^*(\boldsymbol{\theta}) = 2$, and thus an endogenous quota of $q^*(\boldsymbol{\theta}) = 0.5$; in accordance with (4.11), we obtain

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) = \{(2,3,5),(3,2,5),(3,3,4),(4,2,4),(4,3,3)\}$$

(see Figure 9). Now suppose that voter 4 changes her vote to $\tilde{\theta}_4 = (3,2,5)$ while the other voters keep their position. If we denote the resulting profile by $\tilde{\boldsymbol{\theta}}$, we obtain

$$(\tilde{\theta}_i^\ell) \;=\; \begin{pmatrix} 5 & 0 & 5 \\ 0 & 2 & 8 \\ 2 & 6 & 2 \\ 3 & 2 & 5 \end{pmatrix} \quad \textit{and} \quad (\tilde{\theta}_{[k]}^\ell \,|\, \tilde{Q}_{[k]}) \;=\; \begin{pmatrix} 0 & 0 & 2 & | & 2 \\ 2 & 2 & 5 & | & 9 \\ 3 & 2 & 5 & | & 10 \\ 5 & 6 & 8 & | & 19 \end{pmatrix}.$$

Now, since $\tilde{Q}_{[3]} = 3 + 2 + 5 = 10\ (= Q)$, we obtain $k^*(\tilde{\boldsymbol{\theta}}) = 3$, hence an endogenous quota of $q^*(\tilde{\boldsymbol{\theta}}) = 0.75$. Moreover, since $\tilde{Q}_{[k^*(\tilde{\boldsymbol{\theta}})]} = Q$ there is a unique net majority winner, and indeed $\mathrm{CW}^{\mathrm{loc}}(\tilde{\boldsymbol{\theta}}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) = \{(3,2,5)\}$.

As illustrated by the example, Theorem 4 suggests to view the Condorcet solution as the allocations corresponding to a 're-calibrated' coordinate-wise median that maximizes, uniformly across all coordinates, the fraction of agents who can be given their preferred amount of each public good or more.

Using Theorem 4 the complexity of computing the Condorcet solution can be determined as follows. First, one needs to sort n numbers L times. The best sorting algorithms are known to be of order $n \cdot \log n$, hence this yields $L \cdot n \cdot \log n$ computational steps.[14] In addition, one needs to sum L numbers at most n times and compare their sum to the fixed quantity Q. Thus, the determination of the box $\prod_\ell [\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}]$ involves in total a computational complexity of at most

$$L \cdot n \cdot \log n \;+\; L \cdot n \;+\; n$$

single steps. The Condorcet solution corresponding to the profile $\boldsymbol{\theta}$ results from intersecting this box with the resource agenda X.

[14]See, for instance, Knuth (1998).

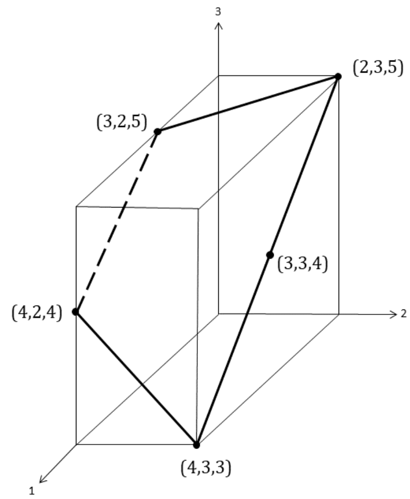

Figure 9: The Condorcet solution $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ *from Example 4*

The simple characterization provided by Theorem 4 allows one to derive further important properties of the Condorcet solution. First, it immediately implies that it respects *coordinate-wise unanimity* in the sense that $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ is contained in the box spanned by the voters' tops. Specifically, we have for all profiles $\boldsymbol{\theta}$,

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \ \subseteq \ boxco(\boldsymbol{\theta}), \tag{4.12}$$

where, for all $Y \subseteq X$, $boxco(Y)$ denotes the smallest box convex set that contains Y.

In fact, Theorem 4 entails much tighter bounds that yield almost uniqueness for large populations, as follows. By (4.11), the ex-ante Condorcet solution is contained in the box $\prod_{\ell=1}^{L}[\theta^{\ell}_{k^*(\boldsymbol{\theta})}, \theta^{\ell}_{k^*(\boldsymbol{\theta})+1}]$. In particular, the 'denser' the support of a profile, the smaller the Condorcet solution. Say that a subset $Y \subseteq X$ is *essentially unique* if

$$\max_{x,y\in Y,\ \ell=1,\ldots,L} |x^{\ell} - y^{\ell}| \leq 1.$$

Thus, a subset of X is essentially unique if every two of its elements differ in each coordinate by at most one unit. Also, say that the support of a profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ is *coordinate-wise connected* if, for each $\ell = 1, ..., L$, the set $\{\theta_i^{\ell}\}_{i=1,...,n}$ forms an interval in $\mathbb{Z}$, i.e. $\{\theta_1^{\ell}, ..., \theta_n^{\ell}\} = [\min_i \theta_i^{\ell}, \max_i \theta_i^{\ell}]$. The following is an immediate corollary of Theorem 4.

Proposition 8. *The Condorcet solution* $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ *is essentially unique whenever* $\boldsymbol{\theta}$ *is coordinate-wise connected.*

If one considers the population preferences as a statistical sample resulting from independent draws from an underlying continuous distribution with connected support, the expected gap $[\theta^{\ell}_{k^*(\boldsymbol{\theta})}, \theta^{\ell}_{k^*(\boldsymbol{\theta})+1}]$ would shrink roughly in inverse proportion to the number of agents n.

Hence the diameter of $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ would likewise shrink in inverse proportion to n. So, heuristically, in such situations one would expect the Condorcet solution to shrink quite rapidly with the number of agents.

4.8 The Condorcet Solution as a Coordinate-Based 'Tukey Median'

At first sight, the proposed solution in the separably convex model, the 1-median, seems to be qualitatively quite different from the proposed solution in the convex model under the similarity hypotheses, the Tukey median. But there is in fact a deep connection, and the ex-ante Condorcet solution can be viewed as a refinement of an appropriate notion of a coordinate-based 'Tukey' median, as follows.

Specifically, for $x \in X$ let $\widetilde{\mathcal{H}}_x$ consider the family of all 'coordinate half-spaces' that contain x, i.e. all half-spaces of the form $H^\ell_{x\uparrow} := \{y \in X : y^\ell \geq x^\ell\}$ or $H^\ell_{x\downarrow} := \{y \in X : y^\ell \leq x^\ell\}$ for $\ell = 1, ..., L$. For every profile $\boldsymbol{\theta}$ and all $x \in X$, denote by

$$\begin{aligned} \widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}) &:= \min_{\ell=1,\dots,L} \#(\boldsymbol{\theta} \cap H^\ell_{x\uparrow}), \\ \widetilde{\mathfrak{d}}{\downarrow}(x;\boldsymbol{\theta}) &:= \min_{\ell=1,\dots,L} \#(\boldsymbol{\theta} \cap H^\ell_{x\downarrow}) \end{aligned}$$

the *upward coordinate depth* of x (at profile $\boldsymbol{\theta}$), and the *downward coordinate depth* of x (at profile $\boldsymbol{\theta}$), respectively; and by $\widetilde{\mathfrak{d}}(x;\boldsymbol{\theta}) := \min\left\{\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}), \widetilde{\mathfrak{d}}{\downarrow}(x;\boldsymbol{\theta})\right\}$ the *(overall) coordinate depth* of x (at profile $\boldsymbol{\theta}$). Define

$$\begin{aligned} \widetilde{T}(\boldsymbol{\theta}) &:= \arg\max_{x\in X} \widetilde{\mathfrak{d}}(x;\boldsymbol{\theta}), \\ \widetilde{T}^*(\boldsymbol{\theta}) &:= \arg\operatorname{lex}\max_{x\in X} \left(\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}), \widetilde{\mathfrak{d}}{\downarrow}(x;\boldsymbol{\theta})\right). \end{aligned}$$

as the *coordinate-based Tukey median* and the *strict coordinate-based Tukey median*, respectively. We have the following result.

Proposition 9. *For all profiles $\boldsymbol{\theta}$,*

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;=\; C_{1-\mathrm{med}}(\boldsymbol{\theta}) \;=\; \widetilde{T}^*(\boldsymbol{\theta}) \;\subseteq\; \widetilde{T}(\boldsymbol{\theta}).$$

(Proof in appendix.)

In the appendix, we show that the inclusion $\widetilde{T}^*(\boldsymbol{\theta}) \subseteq \widetilde{T}(\boldsymbol{\theta})$ is in general strict (see Example 9 in Appendix B).

4.9 Application: Participatory Budgeting

The recent literature on participatory budgeting has focused on the problem of indivisibility of projects, see e.g., Aziz and Shah (2020). One approach to this problem has been considered in Goel *et al.* (2019, Appendix A.1) where divisibility is achieved by a heuristic appeal to 'fractional' implementation of projects. By contrast, here we only assume the existence of *one* divisible good ('money') but allow for the *probabilistic* realization of projects under expected utility preferences. We show that there is always a Condorcet solution in which all projects except at most one are either implemented with certainty, or not at all.

There are $L-1$ projects each of which can either be realized, or not; the realization of project ℓ requires expenditure $c^\ell \geq 0$, $\ell = 1, ..., L-1$; the Lth good is perfectly divisible. Voters have quasilinear utility functions with respect to the Lth good, so that the costs and benefits of all projects can be measured in units of the Lth good which we refer to simply as 'money' in the following. An allocation of expenditure to projects ξ is *feasible* if

$$\xi \in \{0, c^1\} \times ... \times \{0, c^{L-1}\} \times \mathbb{R} \ \text{ and } \ \sum_{\ell=1}^{L} \xi^\ell = 0.$$

Denote by b_i^ℓ the value of project ℓ to voter i, and set $b_i^L = 1$ for all i. For simplicity, we assume that, for each voter, all benefits are pairwise distinct and different from unity; this means that no voter is ever indifferent between two projects and also is never indifferent whether or not a project should be funded. Under this assumption, voter i's (ordinal) preference simply corresponds to a linear ranking of the L different possible uses of the money.

Now let us 'convexify' the problem by considering the set of all probability distributions over the set of feasible allocations, i.e. the agenda

$$X := \left\{ x \in \mathbb{R}^L \mid \sum_{\ell=1}^{L} x^\ell = 0, \ x^\ell \in [0,1] \text{ for all } \ell = 1, ..., L-1 \right\},$$

where x^ℓ is the probability that project ℓ is realized at cost c^ℓ for $\ell = 1, ..., L-1$, and x^L is the (negative of the) expected cost. If voters' preferences over probability distributions have expected utility form they can be represented by the linear utility functions

$$u_i(x) \ = \sum_{\ell=1}^{L} b_i^\ell \cdot x^\ell. \tag{4.13}$$

Note that each voter's top is located at some extreme point of the $L-1$-dimensional polytope X. Denote by $\mathcal{R}_{\text{lin}}$ the set of all preferences representable by linear utility functions of the form (4.13) with pairwise distinct coefficients b_i^ℓ for every i, and by $\boldsymbol{\mathcal{M}}_{\text{lin}} := \cup_{n \in \mathbb{N}} (\mathcal{R}_{\text{lin}})^n$ the (plain) **linear model**. In the appendix we show that $\boldsymbol{\mathcal{M}}_{\text{lin}}$ is a rich submodel of the separably convex model $\boldsymbol{\mathcal{M}}_{\text{sepco}}$ (see Fact B.2); moreover, we have the following result.

Proposition 10. *For every profile* $\boldsymbol{\theta}$*,* $\text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{lin}})$ *contains at least one allocation in which at most one project* ℓ_0 *is funded with probability* $p^{\ell_0} \notin \{0, 1\}$*.*

(Proof in appendix.)

Note that other solutions such as the Tukey median or mean rule would generally yield much more randomization; indeed, the latter would typically randomize over *all* projects. Proposition 10 provides a normative foundation of the analysis in (Goel *et al.*, 2019, Appendix A.1). The 'minimal randomization' solution proposed there of funding those projects that receive the most approvals with certainty until the budget is exhausted, is derived here as the ex-ante Condorcet solution under the assumption of separably convex preferences.

5 Some Pragmatic Considerations

Proposition 9 highlights an important qualitative similarity between the (strict) Tukey median and 1-median rules. From a pragmatic point of view, the 1-median appears to have significant advantages over the Tukey median as an actual voting rule. These include its efficient computability and transparent operation in the light of Theorem 4; by contrast, the Tukey median is notoriously difficult to compute in general (Rousseeuw and Ruts, 1999). In addition, the 1-median has attractive incentive properties that the Tukey median lacks, see Nehring *et al.* (2008); Freeman *et al.* (2021); Nehring and Puppe (2022).

So if one can reasonably expect these rules to give sufficiently similar outcomes, there seem to be good grounds to employ the 1-median rule even when the direct epistemic case for separability is not compelling in itself. For instance, in profiles with maximal Tukey depth $\mathfrak{d}(\boldsymbol{\theta})$ close to $1/2$, the Tukey and the 1-median are close to each other. To see this, let for $\delta > 0$,

$$\widetilde{B}_{\delta}(\boldsymbol{\theta}) \ := \ \left\{x \in X : \tilde{\mathfrak{d}}(x;\boldsymbol{\theta}) \geq \delta\right\},$$

and consider specifically the value $\delta = \mathfrak{d}(\boldsymbol{\theta})$. By construction, we have $T(\boldsymbol{\theta}) \subseteq \widetilde{B}_{\mathfrak{d}(\boldsymbol{\theta})}(\boldsymbol{\theta})$, and by Proposition 9, we have $C_{1-\text{med}}(\boldsymbol{\theta}) \subseteq \widetilde{B}_{\mathfrak{d}(\boldsymbol{\theta})}(\boldsymbol{\theta})$. Thus, both the Tukey median and the 1-median are bounded by the set $\widetilde{B}_{\mathfrak{d}(\boldsymbol{\theta})}(\boldsymbol{\theta})$; in particular, $C_{1-\text{med}}(\boldsymbol{\theta})$ and $T(\boldsymbol{\theta})$ must be close whenever $\widetilde{B}_{\mathfrak{d}(\boldsymbol{\theta})}(\boldsymbol{\theta})$ is small; heuristically, this will be the case if $\mathfrak{d}(\boldsymbol{\theta})$ is close to $1/2$. How this can happen is shown by Caplin and Nalebuff (1988, 1991). Indeed, denoting

$$B_{\delta}(\boldsymbol{\theta}) \ := \ \{x \in X : \mathfrak{d}(x;\boldsymbol{\theta}) \geq \delta\},$$

their main result shows that for $\delta = n/e \approx n \cdot 0.36\%$ the set $B_{\delta}(\boldsymbol{\theta})$ contains both the mean and the Tukey median for distributions with a log concave density on a convex support; since $\widetilde{B}_{\delta}(\boldsymbol{\theta}) \subseteq B_{\delta}(\boldsymbol{\theta})$ it also contains the 1-median. By the triangle inequality, all three allocations will thus be close to each other in these cases.

But, of course, in the absence of these distributional assumptions, the Tukey and 1-median rules can disagree substantially. Indeed, the L_1-distance between the respective allocations can be arbitrarily close to the maximal possible value. Here is an example: consider $Q = 1$ and the following profile $\boldsymbol{\theta}(L)$ consisting of L voters (= the number of dimensions): 2 voters have their top at $(1, 0, 0, ..., 0)$, and for each $\ell = 3, ..., L$ there is exactly one voter such that $\theta^1_\ell = 0$, $\theta^\ell_\ell = 2/L$ and $\theta^k_\ell = 1/L$ for $k \neq 1, \ell$. The unique Tukey median at $\boldsymbol{\theta}(L)$ is $(1, 0, 0, ..., 0)$, while the unique 1-median is $(\frac{1}{L}, \frac{1}{L}, \frac{1}{L}, ..., \frac{1}{L})$ as is easily verified using Theorem 4; the L_1-distance between these allocations is $2 - 2/L$, reflecting the fact that only $1/L$ of total expenditure is allocated to the same goods.

6 Conclusion

In this paper, we have demonstrated that the EAC approach can be fruitfully applied to the budget allocation problem, yielding normatively appealing solutions in cases where the standard Arrovian aggregation methodology fails. Two major question remain in particular: (i) What are the incentive properties of the EAC solution?, and (ii) Is the EAC approach applicable more broadly also in other contexts? Question (i) is addressed in the companion paper Nehring and Puppe (2022). Question (ii) is wide open; we hope that the possibility results presented here may stimulate further research into the merits of the EAC approach and the frugal aggregation framework more broadly.

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Appendix A: Further results, applications and extensions

A.1 The 1-Median with Three Voters

Figure 10 illustrates the Condorcet solution in the separably convex model (aka 1-median) for the case of three voters and $L = 3$. First note that, evidently, for two voters with tops θ and θ', respectively, the 1-median is given by the interval $[\theta, \theta']$. Next, consider the case of three agents with distinct tops θ, θ' and θ'', respectively. In Fig. 10, we fix the two tops θ' and θ'' in generic position, and describe how the Condorcet solution changes when θ moves clockwise 'around' the interval $[\theta', \theta'']$ (with the Condorcet solution marked in red in each case; the depicted shapes of $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ can be easily be verified using Theorems 3 or 4). We note that while the Condorcet solution with three voters is always a triangle (possibly consisting of a single allocation), it can take on a variety of other shapes if there are more than three agents.[15]

A.2 More on Majority Admissibility

Majority admissibility has substantially stronger implications in the separably convex model than in the convex models; nevertheless, it may still not be decisive if $L > 2$. The following example illustrates both of these points.

Example 5. *Consider in the standard simplex with $L = 3$ the top profile $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ with three types of voters none of whom represents an absolute majority; more specifically, suppose that $\theta_1 = (Q/3, 0, 2Q/3)$, $\theta_2 = (0, 2Q/3, Q/3)$ and $\theta_3 = (2Q/3, Q/3, 0)$ (see Figure 11). As is easily verified, the majority admissible set corresponding to this profile is the entire simplex for both the plain convex model and the homogeneous quadratic model. To see this, observe that, for any fixed $x \in X$, the line segments connecting x with each of the three tops never intersect.*

In the separably convex model the majority admissible set is given by the union of three line segments as shown in Fig. 11. This can be verified as follows. Take a point x on the horizontal line segment to the left of the coordinate-wise median $(Q/3, Q/3, Q/3)$ (the 'hub' in Fig. 11). The set $[\theta_1, x]$ lies entirely above the horizontal line segment, the set $[\theta_3, x]$ lies entirely below the horizontal line segment, and we have $[\theta_1, x] \cap [\theta_3, x] = \{x\}$. Since $[\theta_2, x]$ is given by the line segment connecting x and θ_2, we also have $[\theta_1, x] \cap [\theta_2, x] = \{x\}$ and $[\theta_2, x] \cap [\theta_3, x] = \{x\}$. By Lemma 4.1, it follows that x is majority admissible. By symmetry, all points on the other three line segments shown in Fig. 11 are majority admissible. Next, consider any point x such that $x^3 < Q/3$ (i.e. below the horizontal line segment in Fig. 11) and $x^1 > Q/3$ (i.e. to the left of the line segment pointing downwards in Fig. 11). For every sufficiently small $\varepsilon > 0$ we have $(x^1 - \varepsilon, x^2, x^3 + \varepsilon) >^{\mathrm{sepco}}_{\theta_1} x$ and $(x^1 - \varepsilon, x^2, x^3 + \varepsilon) >^{\mathrm{sepco}}_{\theta_2} x$, hence x is not majority admissible. A completely symmetric argument shows the non-admissibility of all other points outside the three line segments shown in Fig. 11.

Finally, observe that in this example,

$$C_{1-\mathrm{med}}(\boldsymbol{\theta}) \;=\; \mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;=\; \{(Q/3, Q/3, Q/3)\}.$$

[15] The website http://www.frugalmajority.de provides an online application to compute and visualize the frugal majority for any number of agents for $L = 3$ and $Q \leq 15$.

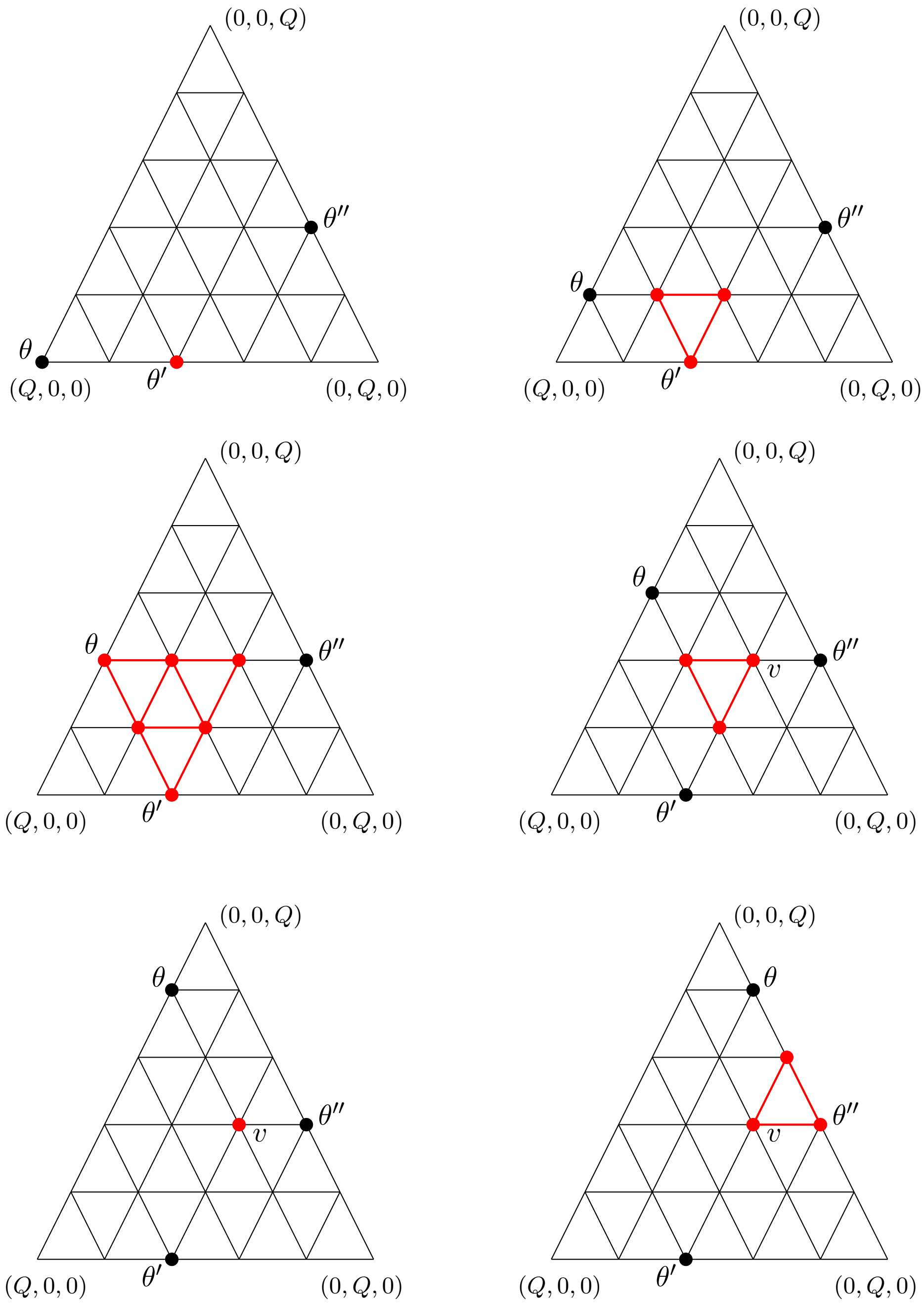

Figure 10: The 1*-median with three agents and* $L = 3$*.*

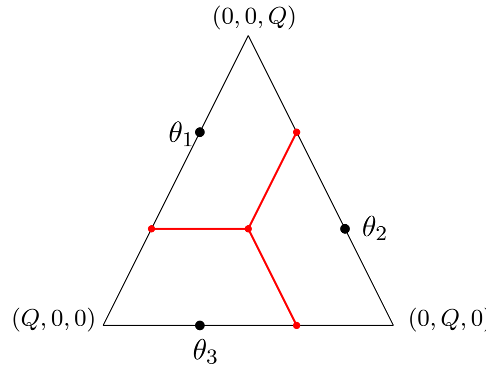

Figure 11: Majority admissibility in the separably convex model

By (2.3), the majority admissible set shrinks as the model becomes more restrictive. While it remains always non-empty in the separably convex model (as is immediate from Theorem 3), it can easily become empty in (non-rich) submodels. Consider, for instance, the plain Euclidean model $\boldsymbol{\mathcal{M}}_{\text{Euclid}}$ consisting of all profiles of ex-post preferences representable by the negative Euclidean distance. For the profile $\boldsymbol{\theta}$ displayed in Example 5 we obtain $\text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{Euclid}}) = \emptyset$. To verify this, note first that clearly no allocation $x \notin co(\boldsymbol{\theta})$ can be majority admissible. On the other hand, for every allocation $x \in co(\boldsymbol{\theta})$, one can choose two distinct tops $\theta_i, \theta_j \in \{\theta_1, \theta_2, \theta_3\}$ such that a movement from x along a circle with center θ_i brings one closer in Euclidean distance to θ_j; by continuity, a position in a neighborhood will then be strictly preferred to x by voters i and j.

By comparison, consider the set of 'L_1-preferences' consisting of all preferences that are representable by the negative L_1-distance to the top allocation $\theta \in X$, i.e. that have a utility function u_θ of the form

$$u_\theta(x) \ = \ -||(x - \theta)||_1.$$

Note that as the plain Euclidean model the plain L_1-model $\boldsymbol{\mathcal{M}}_{L_1}$ consisting of the set of all profiles of L_1-preferences is 'epistemically complete' in the sense that any individual top reveals the entire ex-post preference. As can easily be verified, we have $\text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{L_1}) = \{(Q/3, Q/3, Q/3)\}$ in Example 5 (provided that Q is sufficiently large in order to avoid integer effects). However, the following example shows that also in the L_1-model, the majority admissible set will generally be empty.

Example 6. *Consider in the standard simplex with $L = 3$ the top profile $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ with three types of voters at $\theta_1 = (Q, 0, 0)$, $\theta_2 = (0, Q, 0)$, and $\theta_3 = (0, 0, Q)$, respectively. If θ_1 has the uniquely largest mass but still below 50%, one obtains $\text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{L_1}) = \emptyset$ if Q is sufficiently large; for instance, the allocation $(Q, 0, 0)$ (the unique 1-median) is majority dominated by $(Q/3, Q/3, Q/3)$, which in turn is majority dominated by $(Q/3 + \epsilon, Q/3 + \epsilon, Q/3 - 2\epsilon)$.*

A.3 On the (Non) Existence of Necessary Ex-Post Condorcet Winners

Proposition 1 shows that (under the weak additional condition that the model is copious) the majority admissible alternatives are the *possible* ex-post Condorcet winners. One may wonder if there are interesting, non-trivial conditions on a profile that would guarantee the existence of a *necessary* Condorcet winner, i.e. a Condorcet winner for *every* profile of ex-post preferences compatible with the information $(\boldsymbol{\mathcal{M}}, \boldsymbol{\theta})$. Evidently, an alternative that is the top of more than half of the voters (an 'absolute majority winner') is a necessary Condorcet winner. The following example shows that there is not much hope to weaken the precondition for the existence of necessary Condorcet winners significantly even in simple cases.

Example 7 (Non existence of an ex-post Condorcet winner). *Consider the standard simplex with $L = 3$, $Q = 2$ and the separably convex model $\boldsymbol{\mathcal{M}}_{\text{sepco}}$. Suppose that there are three voters with tops $\theta_1 = (2, 0, 0)$, $\theta_2 = (1, 1, 0)$ and $\theta_3 = (0, 2, 0)$; thus, the corresponding profile $\boldsymbol{\theta}$ is line-like. By Proposition 6(iii), we have $\text{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) = \{\theta_2\}$, hence by Proposition 1, θ_2 is the only candidate for an ex-post Condorcet winner. However, separable convexity is compatible with the preferences $(0, 0, 2) \succ_1 \theta_2$ and $(0, 0, 2) \succ_3 \theta_2$, i.e. θ_2 is not a* necessary *Condorcet winner. (Observe that $(0, 0, 2)$ is not majority admissible since, e.g., $(1, 0, 1) \succ_1 (0, 0, 2)$ and $(1, 0, 1) \succ_2 (0, 0, 2)$.)*

A.4 The 1-Median as 'Frugal Borda' Winners

In this appendix, we show that the Condorcet solution in the separably convex model (aka 1-median) arises naturally also from the perspective of formulating an appropriate 'frugal' version of Borda rule. It is well known that Condorcet consistent voting rules and aggregation rules based on scores in general result in different outcomes (Moulin, 1988a). For instance, even in the simple case of single-peaked preferences on a line and an odd number of voters, the Borda rule may not choose the (ex-post) Condorcet winner (which then always exists and is unique). But in the frugal aggregation model, a natural version of Borda rule coincides with the advocated (ex-ante) Condorcet solution. Intuitively, the reason is that the symmetric treatment of non available preference information entailed by the ex-ante Condorcet approach corresponds to applying Borda rule to *metric* individual preferences that admit an ordinal utility representation in terms of the negative L_1-distance to the top alternative; and for this class of ordinal preferences, Borda's and Condorcet's aggregation methods indeed give the same result.

To make this precise, define for a given a top $\theta \in X$, the *rank* $s_\theta(x) \geq 1$ of an alternative $x \in X$ as follows. A *chain (with respect to $>_\theta^{\text{sepco}}$)* is any subset of X that is totally ordered by the partial order $>_\theta^{\text{sepco}}$. For each $x \in X$, let $s_\theta(x)$ be the maximal cardinality of a chain $Y \ni x$ that has x at its bottom (i.e. $y >_\theta^{\text{sepco}} x$ for all $y \in Y \setminus \{x\}$), so that θ itself uniquely occupies the smallest rank $s_\theta(\theta) = 1$.

For every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every alternative x, let

$$\text{FB}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) := \arg\min_{x \in X} \sum_{i=1}^{n} s_{\theta_i}(x)$$

denote the set of *frugal Borda winners*.[16] It follows from Lemma 4.1, that the rank of an alternative x coincides with the graph distance $d(x,\theta)$ to the top plus one. This immediately entails the following result.

Proposition 11. *For all $\theta \in X$ and all $x \in X$, we have $s_\theta(x) = d(x,\theta)+1$ for the rank derived from the partial order $>_\theta^{\text{sepco}}$. Thus, in particular, for all profiles $\boldsymbol{\theta}$,*

$$\mathrm{FB}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) \;=\; C_{1-\text{med}}(\boldsymbol{\theta}) \;=\; \mathrm{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}).$$

A.5 The 1-Median as Imputed Utilitarian Solution

Theorem 3 represents the Condorcet solution in the separably convex model in terms of a scoring rule, i.e. the minimization of the aggregate L_1-distance to the tops. As noted, this can be interpreted as *utilitarian aggregation* with respect to imputed cardinal utility functions given by the negative L_1-distance. But the economic meaning of these utility functions is unclear, in particular they are *prima facie* only defined on the feasible set X and not on the underlying good space $\mathbb{R}^L$. In this appendix, we show how the utilitarian interpretation can be meaningfully reformulated. Specifically, for each agent with top θ consider the *goal satisfaction* function $v_\theta : \mathbb{R}^L \to \mathbb{R}$ defined by

$$v_\theta(x) := \sum_{\ell=1}^{L} \min\{x^\ell, \theta^\ell\}, \tag{A.1}$$

for all $x \in \mathbb{R}^L$. Evidently, for every $\theta \in X$, $v_\theta(\cdot)$ is an additively separable function which is (weakly) increasing and concave in each component. The term $\min\{x^\ell, \theta^\ell\}$ measures the extent to which the 'goal' θ^ℓ is satisfied in coordinate ℓ by the allocation x.[17] For every profile $\boldsymbol{\theta}$ and all $x \in X$, denote by

$$v_{\boldsymbol{\theta}}(x) \;:=\; \sum_{i=1}^{n} v_{\theta_i}(x),$$

the aggregate goal satisfaction at $x \in X$. The term

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+,$$

where $|x^\ell - \theta^\ell|_+ := \max\{x^\ell - \theta^\ell, 0\}$, can be interpreted as the potential 'waste' of resources at the allocation x from the point of view of an agent with top allocation θ. Noting that, for all $\theta, x \in X$,

$$\sum_{\ell=1}^{L} |x^\ell - \theta^\ell|_+ \;=\; \sum_{\ell=1}^{L} |\theta^\ell - x^\ell|_+ \;=\; d(x,\theta)/2,$$

[16] For different approaches to extending the Borda rule to partial orders, see Young (1974); Cullinan *et al.* (2014).

[17] Note that with monotone preferences, oversatisfaction of the goal in the sense that $x^\ell > \theta^\ell$ does not hurt *per se*; but for a feasible allocation $x \in X$, oversatisfaction in one coordinate is necessarily accompanied by undersatisfaction of the goal in some other coordinate due to the budget constraint.

we obtain for all $x \in X$ and all $\theta_i \in X$,

$$v_{\theta_i}(x) \;=\; \sum_{\ell=1}^{L} x^\ell - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \;=\; Q - \sum_{\ell=1}^{L} |x^\ell - \theta_i^\ell|_+ \tag{A.2}$$

$$\;=\; Q - d(x, \theta_i)/2. \tag{A.3}$$

Equation (A.2) states that, up to a constant, goal satisfaction simply measures aggregate (potential) waste, and (A.3) implies that minimizing aggregate distance of $x \in X$ for a profile $\boldsymbol{\theta}$ of tops in X amounts to maximizing the aggregate goal satisfaction function $v_{\boldsymbol{\theta}}(\cdot)$. Thus, for all profiles $\boldsymbol{\theta}$ in X, the Condorcet solution coincides with the utilitarian maximizers of the individual goal satisfaction functions, i.e. we have the following result.

Proposition 12. *For all profiles* $\boldsymbol{\theta}$*,*

$$C_{1-\mathrm{med}}(\boldsymbol{\theta}) \;=\; \mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;=\; \arg\max_{x \in X} v_{\boldsymbol{\theta}}(x).$$

Within the class of additively separable and concave utility functions, the goal satisfaction functions v_{θ_i} arise naturally from the perspective of the frugal aggregation model, by the following heuristic argument. Consider a (cardinal) differentiable utility function $u(x^1, ..., x^L) = \sum_\ell u^\ell(x^\ell)$ with monotone and concave component functions u^ℓ. In the optimum $\theta \in X$ among all feasible allocations in X, the marginal rates of substitution must all be equal to unity because allocations are defined in terms of expenditure (neglecting any integer problems for simplicity); that is, for the marginal utilities, we obtain $\partial u^\ell(\theta^\ell)/\partial x^\ell = \partial u^k(\theta^k)/\partial x^k$ for all ℓ, k. By the concavity of the component functions, marginal utility is higher below than above the optimum, i.e. for all ℓ, k, all $r < \theta^\ell$ and $\theta^k < s$ we have

$$\frac{\partial u^\ell(r)}{\partial x^\ell} \;\geq\; \frac{\partial u^k(s)}{\partial x^k} \;.$$

Since the only available information in the frugal model is the top θ, an application of the principle of insufficient reason suggests treating all marginals below the top equal to each other, and likewise all marginals above the top. Setting the marginal utilities below the top equal to α and those above the top equal to $\beta < \alpha$, these are affinely equivalent on the feasible set X to the goal-satisfaction utilities defined in (A.1), which correspond in fact to the special case of $\alpha = 1$ and $\beta = 0$. Note that this argument forces the imputed utilities to be non-differentiable but allows them to be strictly monotone.

A.6 The Condorcet Solution Satisfies Reinforcement and the No No-Show Paradox in the Separably Convex Model

The scoring rule representation in terms of the 1-median immediately implies that the Condorcet solution in the separably convex model satisfies the reinforcement condition (3.7). A property closely related to reinforcement is the absence the so-called ‘no-show’ paradox (Fishburn and Brams, 1983; Moulin, 1988b). We show now that the Condorcet solution satisfies a strong version of the avoidance of the no-show paradox in the separably convex model. Specifically, we demonstrate that it can never be harmful for an agent to participate and vote truthfully in the following sense. Given any profile $\boldsymbol{\theta}$, the Condorcet solution $C_{1-\text{med}}(\boldsymbol{\theta} \sqcup \theta_h) = \text{CW}^{\text{loc}}(\boldsymbol{\theta} \sqcup \theta_h, \boldsymbol{\mathcal{M}}_{\text{sepco}})$ resulting from the additional participation of a voter h contains all maximal elements of $C_{1-\text{med}}(\boldsymbol{\theta}) = \text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})$ with respect to $>_{\theta_h}^{\text{sepco}}$; moreover, for every *new* local ex-ante Condorcet winner y resulting from the additional participation of voter h (if any) there exists a maximal element x in $C_{1-\text{med}}(\boldsymbol{\theta})$ with respect to $>_{\theta_h}^{\text{sepco}}$ such that $y >_{\theta_h}^{\text{sepco}} x$.

Formally, for every profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every agent $h \notin \{1, ..., n\}$, denote by $\boldsymbol{\theta} \sqcup \theta_h$ the profile $(\theta_1, ..., \theta_n, \theta_h)$. Moreover, denote by $C_{1-\text{med}}(\boldsymbol{\theta})^h$ the set of all allocations $x \in C_{1-\text{med}}(\boldsymbol{\theta})$ such that $C_{1-\text{med}}(\boldsymbol{\theta}) \cap [x, \theta_h] = \{x\}$. Thus, $C_{1-\text{med}}(\boldsymbol{\theta})^h$ is the subset of *undominated* allocations in $C_{1-\text{med}}(\boldsymbol{\theta})$ from the perspective of an agent with top θ_h (and separably convex preferences). The following result shows that by participating and submitting the top θ_h, an agent is always better off in the sense that (i) the resulting frugal majority winners contain all undominated allocations among the former majority winners, and (ii) every new majority winner (if any) dominates one of these.

Proposition 13. *Consider any profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and any agent $h \notin \{1, ..., n\}$ with top θ_h, then*

$$C_{1-\text{med}}(\boldsymbol{\theta})^h \subseteq C_{1-\text{med}}(\boldsymbol{\theta} \sqcup \theta_h) \subseteq \bigcup_{x \in C_{1-\text{med}}(\boldsymbol{\theta})^h} [x, \theta_h]. \tag{A.4}$$

(Proof in Appendix B.)

Figure 12 illustrates this result. On the left-hand side, the Condorcet solution $C_{1-\text{med}}(\boldsymbol{\theta})$ (without agent h's participation) is marked in red. The right-hand side depicts the Condorcet solution with participation of agent h whose top is at the upper vertex of the red triangle representing $C_{1-\text{med}}(\boldsymbol{\theta} \sqcup \theta_h)$; the subset $C_{1-\text{med}}(\boldsymbol{\theta})^h$ of the undominated elements of $C_{1-\text{med}}(\boldsymbol{\theta})$ is encircled by the black oval. Indeed, from h's perspective, the two allocations discarded by h's participation, θ_1 and θ_2, are strictly worse than their right and left neighbor, respectively; and each of the three local ex-ante Condorcet winners gained by h's participation are strictly preferred by h to at least one element of $C_{1-\text{med}}(\boldsymbol{\theta})^h$.

A.7 Ex-Ante Condorcet vs. Tournament Solutions and the Essential Set

How does the proposed Condorcet solution in the separably convex model depart from standard approaches, and in what way is it superior? Even if one accepts the premise of our present approach to use individual information only about the top alternative plus the common background assumption, i.e. to base the collective decision on the ex-ante majority relation (i.e. in the separably convex model on the inferred partial orders $>_{\theta}^{\text{sepco}}$), one is not forced to use a complete ignorance approach as proposed here, but could try to resort to 'off-the-shelf' solution concepts. First, one could use solution concepts offered by tournament theory, such as the top cycle, or the weak Condorcet winners, see e.g. Laslier (1997), with the underlying tournament given by the (global) ex-ante majority relation defined in (2.5). The problem is with this approach is that the standard solution concepts are either sometimes empty (global Condorcet winners), or frequently very large (the top cycle). To overcome the indeterminacy

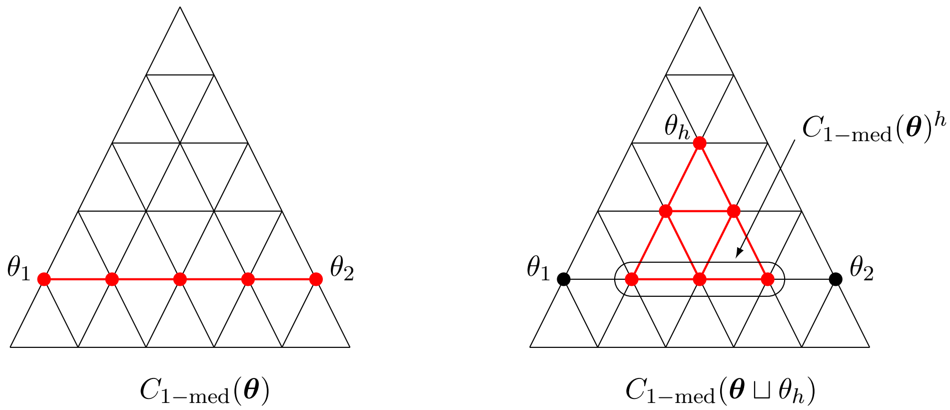

Figure 12: Additional participation of agent h

of the top cycle and related solution concepts one may turn to probabilistic aggregation methods, for instance using the *essential set*, i.e. the support of the maximal lotteries given the tournament $R_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}$ (Brandl *et al.*, 2016). However, this generally also yields additional indeterminacy as compared to frugal majority rule. We illustrate this by means of the following example in which there is no global Condorcet winner, the top cycle is the entire feasible set, but there is a single local ex-ante Condorcet winner.

Example 8. *Suppose that X is given as in (3.1) with $L = 3$, $Q = 3$ and $[q^\ell_-, q^\ell_+] \supseteq [0, 3]$ for all $\ell = 1, 2, 3$. Consider the following profile $\boldsymbol{\theta}$ with seven agents (see Figure 13): $\theta_1 = (1, 1, 1)$, $\theta_2 = \theta_3 = (3, 0, 0)$, $\theta_4 = \theta_5 = (0, 3, 0)$, and $\theta_6 = \theta_7 = (0, 0, 3)$, and fix any rich separably convex model $\mathcal{M}$. Using Lemma 4.1, it is easily verified that $(1, 1, 1) >^{\text{sepco}}_{\theta_i} (0, 1, 2)$ for $i = 1, 2, 3$ and $(0, 1, 2) >^{\text{sepco}}_{\theta_i} (1, 1, 1)$ for $i = 6, 7$, while any ranking between $(1, 1, 1)$ and $(0, 1, 2)$ is compatible with separable convexity for agents $i = 4, 5$. Thus,*

$$(1, 1, 1)P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}(0, 1, 2), \tag{A.5}$$

where $P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}$ is the asymmetric ('strict') part of $R_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}$. Moreover, by Lemma 4.1, we have $(0, 1, 2) >^{\text{sepco}}_{\theta_i} (0, 0, 3)$ for $i = 1, 4, 5$ while any ranking between $(0, 1, 2)$ and $(0, 0, 3)$ is compatible with separable convexity for agents $i = 2.3$; hence, notwithstanding the fact $(0, 0, 3) >^{\text{sepco}}_{\theta_i} (0, 1, 2)$ for $i = 6, 7$, we obtain

$$(0, 1, 2)P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}(0, 0, 3). \tag{A.6}$$

Finally, again using Lemma 4.1 , we obtain

$$(0, 0, 3)P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}(1, 1, 1) \tag{A.7}$$

since agents $i = 6, 7$ have their top at $(0, 0, 3)$ while only agent $i = 1$ has her top at $(1, 1, 1)$ and the ranking between these two allocations is not determined by separable convexity for the other agents $i = 2, 3, 4, 5$. Combining (A.5), (A.6) and (A.7) we thus obtain that both $(1, 1, 1)$ and

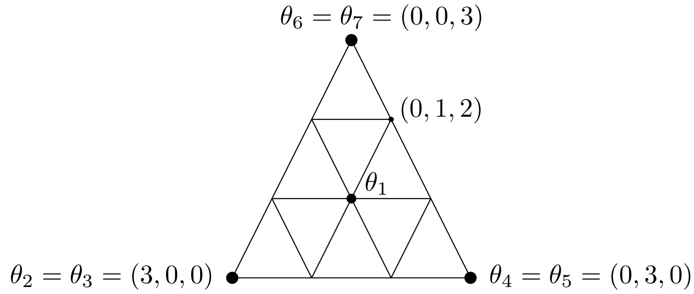

Figure 13: Non-existence of a global ex-ante Condorcet winner

$(0,0,3)$ *are contained in a* $P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}$*-cycle. By a completely symmetric argument, also the allocations* $(3,0,0)$ *and* $(0,3,0)$ *are part of a* $P_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\text{sepco}})}$*-cycle. This implies that the top cycle is indeed the entire set* X*, and that the essential set also contains more than one element.*[18] *However, as is easily verified using Theorem 4 above, the (local) Condorcet solution for this profile is the single allocation*

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) \;=\; \{(1,1,1)\}.$$

And indeed, in view of the symmetry of the profile, the allocation $(1,1,1)$ seems to be the clear optimal choice, and it is questionable if there is any identifiable advantage from including other allocations in the choice set. Our diagnosis is that not all majority comparisons are equally important, and arguably *local* comparisons contain all useful information. By treating local and non-local majority comparisons on par, standard approaches add noise which may lead to inferior recommendations.

A.8 The Local Ex-ante Condorcet Winners in the Continuous Case

Our local majoritarian foundation of the Condorcet solution (aka 1-median) in the separably convex model can be re-formulated for continuous resource agendas $X \subset \mathbb{R}^L$, as follows. A preference order $\succcurlyeq$ on X is *separably convex* if the following condition is satisfied: whenever a marginal transfer from good j to good k at allocation $x \in X$ makes an agent worse off (keeping the allocation fixed otherwise), then so does the same transfer at any allocation that has at most the amount x^j of good j and at least the amount x^k of good k. Given a profile of tops $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ on X, define the symmetric binary relation $\Gamma_{\boldsymbol{\theta}}$ on X by $x\Gamma_{\boldsymbol{\theta}}y$ if (i) x and y differ in exactly two coordinates, i.e. $x \neq y$ and for some distinct $j,k \in \{1,...,L\}$, $x^\ell = y^\ell$ for all $\ell \neq j,k$, and (ii) for no $i = 1,...,n$, $\min\{x^j,y^j\} < \theta_i^j < \max\{x^j,y^j\}$ or $\min\{x^k,y^k\} < \theta_i^k < \max\{x^k,y^k\}$. Thus, if $x\Gamma_{\boldsymbol{\theta}}y$ then x and y are 'neighbors' in the sense that they differ only in two coordinates and no top lies strictly between them in these two coordinates; geometrically, condition (ii) means that no top lies in the 'stripe' between x^j

[18] The essential set is single-valued if and only if this single element constitutes a Condorcet winner, see Brandl *et al.* (2016).

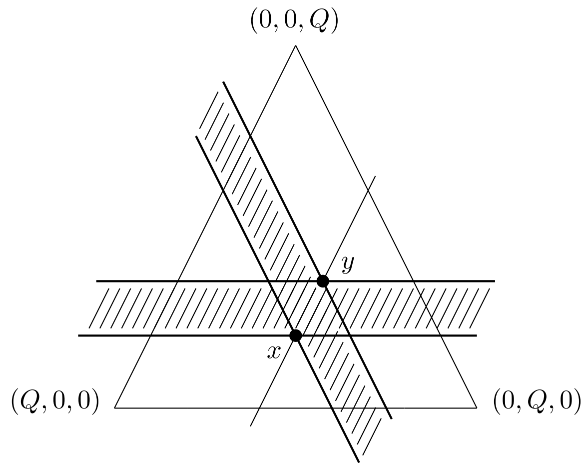

Figure 14: The binary relation $\Gamma_{\boldsymbol{\theta}}$

and y^j parallel to the j-axis, and no top lies in the 'stripe' between x^j and y^j parallel to the j-axis, see Figure 14. Observe that for every profile with finite support, and for all j, k, this condition is satisfied whenever x and y are sufficiently close to each other.

Similar to Section 4.4 above, define the local ex-ante majority relation now profile by profile with respect to the relation $\Gamma_{\boldsymbol{\theta}}$, i.e. for all $x, y \in X$,

$$xR^{\mathrm{loc}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{sepco}})}y \ :\Longleftrightarrow\ \left[xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{sepco}})}y \ \text{ and } \ x\Gamma_{\boldsymbol{\theta}}y\right]$$

and let

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \ = \ \{x \in X \mid \text{for no } y \in X,\ yP^{\mathrm{loc}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{sepco}})}x\},$$

as before (for simplicity, we do not notationally distinguish corresponding concepts in the discrete and the continuous case).

Theorem 3′ *For all profiles* $\boldsymbol{\theta}$*, the set* $\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}_{\mathrm{sepco}})$ *is non-empty, box-convex and coincides with* $C_{1-\mathrm{med}}(\boldsymbol{\theta})$ *(the latter set being defined exactly as in the discrete case).*

(Proof in Appendix B.)

The characterization of the Condorcet solution provided in Theorem 4 continues to hold without change for profiles with finite support. In fact, the endogenous quota characterization provided by Theorem 4 can be used to define the Condorcet solution for general distributions of agent's tops in a straightforward manner. In the case of atomless and continuous distributions it always yields a unique solution, as follows. For each $\ell = 1, ..., L$, and all $t \in [0, 1]$, denote by $\xi^\ell(t)$ the cumulative distribution of the tops in coordinate ℓ, i.e. $\xi^\ell(t) = r$ if and only if the fraction t of agents' tops has at most the amount r in coordinate ℓ; evidently, $\xi^\ell(\cdot)$ is an increasing function for all ℓ. Let $Q(t) := \sum_{\ell=1}^{L} \xi^\ell(t)$ which is clearly also increasing. If the underlying distribution $\boldsymbol{\theta}$ of tops is atomless, $Q(\cdot)$ is in fact *strictly* increasing and continuous on $[0,1]$ with $Q(0) < Q < Q(1)$. By the intermediate value theorem, there exists exactly one $t^* \in (0,1)$ such that $Q(t^*) = Q$; then

$$\mathrm{CW}^{\mathrm{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\mathrm{sepco}}) \;=\; \{(\xi^1(t^*), ..., \xi^L(t^*))\},$$

i.e. the allocation $(\xi^1(t^*), ..., \xi^L(t^*))$ is the unique frugal majority winner.

Appendix B: Remaining Proofs

Proof of Proposition 1. By contraposition, let $x \notin \mathrm{MA}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$; we show that x cannot be an ex-post Condorcet winner. By definition, there exist $y \in X$, such that $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x) > m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y)$; in particular, for all $\succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$, $\{i : y \succ_i x\} > \{i : x \succ_i y\}$. Hence, x is not a Condorcet winner at any profile $\succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$.

Conversely, let $x \in \mathrm{MA}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$, i.e. for no $y \in X$, $yP^{\mathrm{nec}}_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$, or in other words, for all $y \in X$,

$$m^+_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y) \;\geq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x).$$

For all $y \in X \setminus \{x\}$, choose $\succcurlyeq^y \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ such that $\#\{i : x \succ_i^y y\}$ is maximized; in particular, for such a profile we have $\#\{i : x \succ_i^y y\} \geq \#\{i : y \succ_i^y x\}$ (since otherwise, y would be a necessary majority winner gainst x). Denote by $J^y := \{i : x \succ_i^y y\}$, and by $J := \cup_{y \neq x} J^y$. By the richness condition (2.4), there exists $\succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ such that for all $i \in J$, $x \succ_i y$ for all $y \neq x$. Since $\#J \geq \#J^y$ for all $y \neq x$, we have

$$\#\{i : x \succ_i y\} \;\geq\; \#\{i : y \succ_i x\}$$

for all $y \neq x$, i.e. x is an ex-post Condorcet winner at $\succcurlyeq \in \Omega_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$. □

Proof of Proposition 2. The proof is straightforward and the argument is well-known. □

Proof of Proposition 3. Under convexity of preferences a necessary and sufficient condition for $x \succ_i y$ to hold for a voter with top θ_i is that $x \neq y$ and x is on a straight line connecting y and θ_1. This directly implies the statement of the proposition. □

The following fact generalizes Proposition 2 to embedded one dimensional subspaces.

Fact B.1. *Let $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{\mathrm{co}}$ (plain convex model) or $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}^{reg}_{\mathrm{co}}$ (regularized convex model), and let $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ a profile of collinear tops. Specifically, suppose that the tops θ_i are ordered along a Euclidean straight line $L \subseteq X$ such that, for all i, $\theta_i = \theta_1 + \lambda_i \cdot r$, for some $r \in \mathbb{R}^L$ and λ_i such that $0 \leq \lambda_i \leq \lambda_{i'}$ whenever $i \leq i'$. Then, $\mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \cap L = \{\theta_{n+1)/2}\}$ if n is odd, and $\mathrm{MA}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \cap L = [\theta_{n/2}, \theta_{n/2+1}] \cap L$ if n is even.*

Proof. Evidently, the 'median top(s) within L' are necessary majority winners against all other alternatives in L in either convex model; moreover, no alternative outside L is a necessary majority winner against any of the 'median top(s) within L.' □

Proof of Proposition 4. Consider any $x \neq \theta_{i^*}$; by assumption, there is at most one $\theta_j \neq \theta_{i^*}$ on the straight line through x and θ_{i^*}. By the observation in the proof of Proposition 3, and since θ_{i^*} has largest popular support, this implies $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(\theta_{i^*}, x) \geq m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, \theta_{i^*})$, i.e. θ_{i^*} is an ex-ante majority winner against x; if θ_{i^*} has uniquely largest popular support, we even have $m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(\theta_{i^*}, x) > m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x, \theta_{i^*})$. Since x was chosen arbitrarily, the result follows. □

Proof of Fact 3.3. By definition, a profile of quadratic preferences is homogeneous if the marginal distribution $\mu|_{\theta_i}$ over the quadratic forms is independent of the top θ_i. This means that the profile is a product measure over $X \times \boldsymbol{\mathcal{Q}}$, where $\boldsymbol{\mathcal{Q}}$ is the set of all quadratic forms. But this also means that also the marginal distribution over the tops is independent of the quadratic form $\mathcal{Q}$. This implies at once that, for all distinct $x, y \in X$,

$$m^-_{(\boldsymbol{\theta},\widehat{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(x, y) \ = \ m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(x, y),$$

hence the statement of Fact 3.3. □

We prove Theorems 1 and 2 in reverse order. For the proof of Theorems 2 we need a series of lemmata. Denote the *relative Tukey depth of x with respect to y* by

$$m^-_{(\boldsymbol{\theta},\text{Tuk})}(x, y) := \min_{H \in \mathcal{H}_x,\, y \notin H} \#(\boldsymbol{\theta} \cap H),$$

and $x R_{(\boldsymbol{\theta},\text{Tuk})} y :\Leftrightarrow m^-_{(\boldsymbol{\theta},\text{Tuk})}(x, y) \geq m^-_{(\boldsymbol{\theta},\text{Tuk})}(y, x)$, as well as

$$\text{CW}(\boldsymbol{\theta}, \text{Tuk}) \ := \ \{x \in X |\ \text{for no } y \in X,\ y\, P_{(\boldsymbol{\theta},\text{Tuk})} x\}.$$

Lemma B.1. *For all distinct $x, y \in X$ and all profiles $\boldsymbol{\theta}$, $R_{(\boldsymbol{\theta},\text{SH})} = R_{(\boldsymbol{\theta},\text{Tuk})}$, cf. (3.5).*

Proof. We show that, for all $x, y \in X$,

$$\widehat{m}^-_{(\boldsymbol{\theta},\text{SH})}(x, y) \ = \ \min_{H \in \mathcal{H}_x,\, y \notin H} \#(\boldsymbol{\theta} \cap H) \ = \ m^-_{(\boldsymbol{\theta},\text{Tuk})}(x, y); \tag{B.1}$$

from this, the statement of Lemma B.1 is immediate. Let H^0 be a half-space with $x \in H^0$ and $y \notin H^0$ for which $\#(\boldsymbol{\theta} \cap H)$ is minimal. Choose for all tops $\theta_i \in H^0$ a convex preference $\succcurlyeq_i$ such that $x \succ_i y$, and for all tops $\theta_i \notin H^0$ a convex preference $\succcurlyeq_i$ such that $y \succ_i x$. Evidently, the constructed profile satisfies the similarity hypothesis (without any 'misclassification'); hence, the term on the left-hand side of equation (B.1) cannot be strictly larger than the term on the right-hand side.

Conversely, consider a profile $\succcurlyeq$ of convex preferences satisfying the similarity hypothesis, and assume that it attains the minimal number of x-supporters among all those. By the minimality, we may assume without loss of generality that $\#\{i : x \succ_i y\} \leq \#\{i : y \succ_i x\}$. Since the misclassification is bounded by condition (ii) in the similarity hypothesis, the number of x-supporters in a profile satisfying the similarity hypothesis with respect to the linear classification H_x, H_y cannot be smaller than $\#(\boldsymbol{\theta} \cap H_x)$. This shows the equality in (B.1). □

In particular, by Lemma B.1 we have $\text{CW}(\boldsymbol{\theta}, \text{Tuk}) = \text{CW}(\boldsymbol{\theta}, \text{SH})$ and we now show that this set coincides with the strict Tukey median.

Lemma B.2. *For all $x, y \in X$, $\mathfrak{d}(x;\boldsymbol{\theta}) > \mathfrak{d}(y;\boldsymbol{\theta})$ implies $xP_{(\boldsymbol{\theta},\mathrm{Tuk})}y$.*

Proof. Let us henceforth abbreviate and write $\boldsymbol{\theta}(H) := \#(\boldsymbol{\theta} \cap H)$. By assumption there exists a half-space H containing y with $\boldsymbol{\theta}(H) < \mathfrak{d}(x;\boldsymbol{\theta})$, hence in particular $x \notin H$. Thus,

$$m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) \;\geq\; \mathfrak{d}(x;\boldsymbol{\theta}) \;>\; \boldsymbol{\theta}(H) \;\geq\; m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(y,x).$$

□

Note that Lemma B.2 implies $\mathrm{CW}(\boldsymbol{\theta},\mathrm{Tuk}) = \mathrm{CW}(\boldsymbol{\theta},\mathrm{SH}) \subseteq T(\boldsymbol{\theta})$.

Lemma B.3. *For all $x, y \in X$ with $\mathfrak{d}(x;\boldsymbol{\theta}) = \mathfrak{d}(y;\boldsymbol{\theta}) =: \alpha$, one has $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) = \alpha$ or $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(y,x) = \alpha$.*

Proof. Let $H \ni x$ be such that $\boldsymbol{\theta}(H) = \alpha$; without loss of generality assume that x is on the boundary of H. Note that H cannot have positive mass at the boundary (except possibly at x itself); indeed, otherwise a small rotation around x would lower $\boldsymbol{\theta}(H)$ which is already minimal.

If $y \notin H$, then $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) = \alpha$. If $y \in H$ one can find H' 'close' to H such that $\boldsymbol{\theta}(H') = \boldsymbol{\theta}(H)$, $y \in H$ and $x \notin H'$. Indeed, if y is in the interior of H, move the boundary of H slightly parallel towards y, and if y is on the boundary of H rotate the boundary slightly to H' so that x 'drops out.' Thus, $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(y,x) = \alpha$. □

Lemma B.4. *For all $x, y \in X$ with $\mathfrak{d}(x;\boldsymbol{\theta}) = \mathfrak{d}(y;\boldsymbol{\theta}) =: \alpha$,*

$$xP_{(\boldsymbol{\theta},\mathrm{Tuk})}y \iff m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) > \alpha \iff \mathcal{H}^{\alpha}_{x} \subsetneq \mathcal{H}^{\alpha}_{y},$$

where $\mathcal{H}^{\alpha}_{x} := \{H \in \mathcal{H} \mid x \in H \text{ and } \boldsymbol{\theta}(H) = \alpha\}$.

Proof. In view of Lemma B.3, we need to show that $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) > \alpha \iff \mathcal{H}^{\alpha}_{x} \subsetneq \mathcal{H}^{\alpha}_{y}$. If $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) > \alpha$, then there exists a half-space H such that $y \in H$, $x \notin H$ and $\boldsymbol{\theta}(H) = \alpha$, but there does not exist a half-space H such that $x \in H$, $y \notin H$ and $\boldsymbol{\theta}(H) = \alpha$; by the latter, $\mathcal{H}^{\alpha}_{x} \subseteq \mathcal{H}^{\alpha}_{y}$, hence by the former in fact $\mathcal{H}^{\alpha}_{x} \subsetneq \mathcal{H}^{\alpha}_{y}$.

Conversely, if $\mathcal{H}^{\alpha}_{x} \subsetneq \mathcal{H}^{\alpha}_{y}$, there does not exist a half-space H such that $x \in H$, $y \notin H$ and $\boldsymbol{\theta}(H) = \alpha$, hence $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(x,y) > \alpha$. □

Observe that, by Lemma B.4 the relation $P_{(\boldsymbol{\theta},\mathrm{Tuk})}$ is a strict partial order when restricted to any depth level set. Now consider the Tukey median $T(\boldsymbol{\theta})$, i.e. the depth level set with maximal depth, and denote by $\widetilde{L}_x(\boldsymbol{\theta}) := \{y \in T(\boldsymbol{\theta}) \mid xR_{(\boldsymbol{\theta},\mathrm{Tuk})}y\} \setminus \{x\}$ (i.e. the lower contour set of x with respect to $R_{(\boldsymbol{\theta},\mathrm{Tuk})}$ minus the alternative x itself).

Lemma B.5. *For all $x \in T(\boldsymbol{\theta})$, the sets $\widetilde{L}_x(\boldsymbol{\theta})$ are relative open in $T(\boldsymbol{\theta})$.*

Proof. Consider any pair $x, y \in T(\boldsymbol{\theta})$ such that $xR_{(\boldsymbol{\theta},\mathrm{Tuk})}y$ and $x \neq y$, and let α^* be the maximal Tukey depth. By Lemmas B.3 and B.4, we have $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(y,x) = \alpha^*$. Thus, there exists a half-space H with $\boldsymbol{\theta}(H) = \alpha^*$, $y \in H$, $x \notin H$, and without loss of generality we may assume that the boundary of H has mass zero. Therefore, by moving H slightly (say, in a parallel way) towards x, we obtain $m^{-}_{(\boldsymbol{\theta},\mathrm{Tuk})}(y',x) = \alpha^*$, and hence $xR_{(\boldsymbol{\theta},\mathrm{Tuk})}y'$, for all y' in a small neighborhood of y. This shows that $\widetilde{L}_x(\boldsymbol{\theta})$ is relative open in $T(\boldsymbol{\theta})$. □

Proof of Theorem 2. By Lemma B.4, we have $\mathrm{CW}(\boldsymbol{\theta}, \mathrm{Tuk}) = \mathrm{CW}(\boldsymbol{\theta}, \mathrm{SH}) = T^*(\boldsymbol{\theta})$. Thus, it only remains to be shown that this set is indeed non-empty. Let

$$\widetilde{U}_x(\boldsymbol{\theta}) := \{y \in T(\boldsymbol{\theta}) \mid yP_{(\boldsymbol{\theta},\mathrm{Tuk})}x\} \cup \{x\}$$

Consider chains of such 'upper contour' sets, i.e. subsets $\mathcal{C} \subseteq \{\widetilde{U}_x(\boldsymbol{\theta}) \mid x \in T(\boldsymbol{\theta})\}$ totally ordered by set inclusion, and denote by $\mathcal{U}$ the family of all such chains partially ordered by set inclusion. By Zorn's Lemma, there exists a maximal element in $\mathcal{U}$, i.e. a maximal chain $\mathcal{C}^*$. By Lemma B.5, the elements of $\mathcal{C}^*$ are relative closed, and since $T(\boldsymbol{\theta})$ is bounded, they are in fact (relative) compact. Hence, the intersection $\cap\mathcal{C}^*$ is non-empty, and by maximality of $\mathcal{C}^*$ it consists of a unique element, say $\widetilde{U}_{x^*}(\boldsymbol{\theta})$. By construction, $x^* \in \mathrm{CW}(\boldsymbol{\theta}, \mathrm{Tuk})$, in particular $\mathrm{CW}(\boldsymbol{\theta}, \mathrm{Tuk}) = T^*(\boldsymbol{\theta})$ is non-empty. $\square$

We now turn to the proof of Theorem 1; again, we need some preliminary results.

Lemma B.6. *Let $\boldsymbol{\mathcal{M}} \subseteq \boldsymbol{\mathcal{M}}_{\mathrm{co}}$ be any model of convex preferences, and suppose that $z \in (x, y]$, then*

$$\left[m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y) \;\geq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,z)\right] \quad \textit{and} \quad \left[m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(z,x)\right]. \tag{B.2}$$

In particular, $yP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$ implies $zP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$, and $xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}z$ implies $xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}y$.

Proof. Recalling our slightly stronger than standard notion of 'convex preference,' we have for every convex preference $\succcurlyeq$,

$$\begin{aligned} x \succ z \;&\Longrightarrow\; x \succ y, \text{ and} \\ y \succ x \;&\Longrightarrow\; z \succ x. \end{aligned}$$

In particular, every preference admissible in $\boldsymbol{\mathcal{M}}$ that supports x against z also supports x against y; this shows the first part of (B.2). Similarly, every preference admissible in $\boldsymbol{\mathcal{M}}$ that supports y against x also supports z against x which shows the second part of (B.2). If $yP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$, we thus have

$$m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,z) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y) \;<\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(z,x),$$

i.e. $zP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$. Similarly, if $xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}z$, we have

$$m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(y,x) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(z,x) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,z) \;\leq\; m^-_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}(x,y),$$

i.e. $xR_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}y$. $\square$

Define the 'local' (strict) net majority relation $Q_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}$ as follows. For all $x, y \in X$,

$$y\,Q_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x \;:\Longleftrightarrow\; zP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x \;\text{ for all } z \in (x,y], \tag{B.3}$$

and let

$$\mathrm{LCW}(\boldsymbol{\theta},\boldsymbol{\mathcal{M}}) \;:=\; \{x \in X \mid \text{for no } y \in X,\; y\,Q_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x\}. \tag{B.4}$$

Lemma B.7. *Let $\boldsymbol{\mathcal{M}} \subseteq \boldsymbol{\mathcal{M}}_{\text{co}}$; for all $\boldsymbol{\theta}$,*

$$\text{LCW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \;=\; \text{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}).$$

Proof. Evidently, $\text{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}) \subseteq \text{LCW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}})$. The converse is shown by contraposition. Suppose that $x \notin \text{CW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}})$, i.e., $yP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$ for some $y \in X$. By Lemma B.6, we obtain $zP_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$ for all $z \in (x, y]$, i.e. $y\,Q_{(\boldsymbol{\theta},\boldsymbol{\mathcal{M}})}x$; thus, $x \notin \text{LCW}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}})$. □

Lemma B.8. *For all $\boldsymbol{\theta}$,*

$$\text{CW}(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}}) \;=\; \text{CW}(\boldsymbol{\theta}, \text{Tuk}).$$

Proof. Let $x^* \in \text{CW}(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})$, i.e. $x^*R_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}y$ for all $y \in X$. By contradiction, assume that $yP_{(\boldsymbol{\theta},\text{Tuk})}x^*$ for some $y \in X$, i.e.

$$m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y) < m^-_{(\boldsymbol{\theta},\text{Tuk})}(y, x^*). \tag{B.5}$$

Let $H^0 \in \mathcal{H}_{x^*}$ be a Euclidean half-space that separates x^* from y and that minimizes $\#(\boldsymbol{\theta} \cap H)$ among all such half-spaces. Without loss of generality, we may assume that $x^* \in \partial H^0$ and that $\partial H^0 \cap \boldsymbol{\theta} \subseteq \{x^*\}$ (the latter by the fact that $\boldsymbol{\theta}$ is made up by a finite set of points). Therefore, we my shift H^0 slightly along the straight line S connecting y and x^* in a parallel way towards y to $\tilde{H}^0$ while keeping the mass with respect to $\boldsymbol{\theta}$ constant, i.e. such that $\#(\boldsymbol{\theta} \cap H^0) = \#(\boldsymbol{\theta} \cap \tilde{H}^0) = m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y)$. Consider the intersection w of S with $\partial\tilde{H}^0$ and the point z on S such that w is the midpoint between w and x^* (see Figure 15). By Fact 3.2 we have $m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(x^*, z) = \#(\boldsymbol{\theta} \cap \tilde{H}^0) = m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y)$. Moroever, we evidently also have $m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y) = m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, z)$, and $m^-_{(\boldsymbol{\theta},\text{Tuk})}(z, x^*) \geq m^-_{(\boldsymbol{\theta},\text{Tuk})}(y, x^*)$. Thus, using (B.5) and the fact that, for all $w, v \in X$, $m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(w, v) \geq m^-_{(\boldsymbol{\theta},\text{Tuk})}(w, v)$, we obtain,

$$\begin{aligned} m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(z, x^*) \geq\; & m^-_{(\boldsymbol{\theta},\text{Tuk})}(z, x^*) \;\geq\; m^-_{(\boldsymbol{\theta},\text{Tuk})}(y, x^*) \\ & > \; m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y) \;=\; m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, z) \;=\; m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}(x^*, z). \end{aligned}$$

i.e. $zP_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})}x^*$ in contradiction to the initial assumption that $x^* \in \text{CW}(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\text{quad}})$.

Conversely, let $x^* \in \text{CW}(\boldsymbol{\theta}, \text{Tuk})$, i.e. $x^*R_{(\boldsymbol{\theta},\text{Tuk})}x$ for all $x \in X$. Consider any fixed $y \in X$ distinct from x^*. Let $H^0 \in \mathcal{H}_{x^*}$ be a Euclidean half-space that separates x^* from y and that minimizes $\#(\boldsymbol{\theta} \cap H)$ among all such half-spaces. Without loss of generality, we may assume that $x^* \in \partial H^0$ and that $\partial H^0 \cap \boldsymbol{\theta} \subseteq \{x^*\}$ (the latter by the fact that $\boldsymbol{\theta}$ is made up by a finite set of points). Therefore, we my shift H^0 slightly along the straight line S connecting y and x^* in a parallel way towards y to $\tilde{H}^0$ while keeping the mass with respect to $\boldsymbol{\theta}$ constant, i.e. such that $\#(\boldsymbol{\theta} \cap H^0) = \#(\boldsymbol{\theta} \cap \tilde{H}^0) = m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y)$. Consider the intersection w of S with $\partial\tilde{H}^0$ and the point z on S such that w is the midpoint between w and x^* (as in Fig. 15 again). We clearly have $m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, y) = m^-_{(\boldsymbol{\theta},\text{Tuk})}(x^*, w) = \#(\boldsymbol{\theta} \cap \tilde{H}^0)$. Let H^1 be a Euclidean half-space containing w that minimizes $\#(\boldsymbol{\theta} \cap H)$ among all half-spaces that contain w and that separate

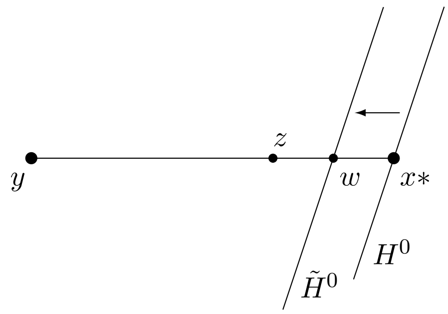

Figure 15: Argument from Lemma B.8

w from x^*, i.e. $m^-_{(\boldsymbol{\theta},\mathrm{Tuk})}(w, x^*) = \#(\boldsymbol{\theta} \cap H^1)$. Without loss of generality, we may assume that $w \in \partial H^1$. Since $x^* \in \mathrm{CW}(\boldsymbol{\theta}, \mathrm{Tuk})$, we have $x^* R_{(\boldsymbol{\theta},\mathrm{Tuk})} w$, i.e.

$$\#(\boldsymbol{\theta} \cap \tilde{H}^0) \;=\; m^-_{(\boldsymbol{\theta},\mathrm{Tuk})}(x^*, w) \;\geq\; m^-_{(\boldsymbol{\theta},\mathrm{Tuk})}(w, x^*) \;=\; \#(\boldsymbol{\theta} \cap H^1).$$

Since w is the midpoint between z and x^*, we obtain using Fact 3.2 that

$$m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})}(x^*, z) \;=\; \#(\boldsymbol{\theta} \cap \tilde{H}^0) \;\geq\; \#(\boldsymbol{\theta} \cap H^1) \;=\; m^-_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})}(z, x^*),$$

i.e. $x^* R_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})} z$ for some $y \in X$. By Lemma B.6 this implies $x^* R_{(\boldsymbol{\theta},\overline{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})} y$. Since y was arbitrarily chosen, we thus finally obtain $x^* \in \mathrm{CW}(\boldsymbol{\theta}, \overline{\boldsymbol{\mathcal{M}}}_{\mathrm{quad}})$. □

Proof of Theorem 1. The proof follows from combining Fact 3.3, Lemmas B.1, B.8 and Theorem 2. □

Proof of Fact 3.4. The statement follows at once from the fact that every affine mapping with full rank transforms any Euclidean half-space again into a Euclidean half-space. □

Proof of Proposition 5. By contradiction, suppose that $C(\cdot)$ is a solution satisfying upper hemicontinuity, the qualified affine equivariance condition (3.6), and majority admissibility with respect to the plain convex model. Take any profile $\boldsymbol{\theta}^0$ with tops $(\theta_1^0, ..., \theta_n^0)$ in general position and n odd, and consider $x \in C(\boldsymbol{\theta}^0)$. If $x \in co(\boldsymbol{\theta}^0)$, there are coefficients $a_i \in \mathbb{R}$ such that $x = \sum_{i=1}^n a_i \theta_i^0$. By the affine equivariance condition (3.6), we obtain $\sum_{i=1}^n a_i \theta_i \in C(\boldsymbol{\theta})$ for all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ in general position. By upper hemicontinuity, we indeed obtain $\sum_{i=1}^n a_i \theta_i \in C(\boldsymbol{\theta})$ for *all* profiles. However, this contradicts majority admissibility if the profile $\boldsymbol{\theta}$ is collinear, since $\sum_{i=1}^n a_i \theta_i$ will in general differ from the median top. □

Fact B.2. *The models $\boldsymbol{\mathcal{M}}_{\mathrm{lin}}$ and $\boldsymbol{\mathcal{M}}_{\mathrm{addco}}$ are both rich submodels of the plain separably convex model $\boldsymbol{\mathcal{M}}_{\mathrm{sepco}}$.*

Proof. First, we show that any preference $\succcurlyeq \in \boldsymbol{\mathcal{M}}_{\mathrm{add}}$ is separably convex; since $\boldsymbol{\mathcal{M}}_{\mathrm{lin}} \subseteq \boldsymbol{\mathcal{M}}_{\mathrm{add}}$ the same conclusion holds for the linear as well. Thus, let $\succcurlyeq$ be represented as in (4.2) by an additively separable utility function $u(x) = \sum_\ell u^\ell(x^\ell)$ with strictly increasing and concave component functions $u^\ell : \mathbb{R} \to \mathbb{R}$. In fact, the separable convexity follows from the concavity alone, no monotonicity condition on the functions u^ℓ is needed. Indeed, suppose that $x \succ x_{(kj)}$ and $y^j \leq x^j$ as well as $y^k \geq x^k$; since the allocations x and $x_{(kj)}$ differ only in coordinates j and k, we have $u(x) > u(x_{(kj)})$ if and only if $u^j(x^j) - u^j(x^j - 1) > u^k(x^k + 1) - u^k(x^k)$. By the concavity of $u^k(\cdot)$ and $u^j(\cdot)$ we obtain

$$\left[u^j(y^j) - u^j(y^j - 1)\right] \geq \left[u^j(x^j) - u^j(x^j - 1)\right] > \left[(u^k(x^k + 1) - u^k(x^k)\right] \geq \left[u^k(y^k + 1) - u^k(y^k)\right],$$

and hence $u(y) > u(y_{(kj)})$ as desired.

Next, we show that $\boldsymbol{\mathcal{M}}_{\text{add}}$ is indeed a *rich* separably convex model. Thus, consider $x, y, \theta \in X$ such that $x \notin [\theta, y]$. We will show that there exists $\succcurlyeq \in \boldsymbol{\mathcal{M}}_{\text{add}}$ with top θ such that $y \succ x$. This in fact not only shows that $\boldsymbol{\mathcal{M}}_{\text{add}}$ is a rich separably convex model but, in addition, that every allocation can be the top of a preference order in $\boldsymbol{\mathcal{M}}_{\text{add}}$. The statement is trivial if $\theta = y$; thus, assume henceforth that $\theta \neq y$. In the following, we explicitly construct appropriate strictly increasing and strictly concave functions $u^\ell : X^\ell \to \mathbb{R}$ for $\ell = 1, ..., L$, where X^ℓ is the projection of X to coordinate ℓ. First observe that it is clearly possible, for any given $\theta^\ell \in X^\ell$ and any $\epsilon > 0$, to slightly 'perturb' the identity function $f(x^\ell) = x^\ell$ to a strictly concave and strictly increasing function $\tilde{f}$ such that the difference $\tilde{f}(\theta^\ell) - \theta^\ell$ is strictly larger that $\tilde{f}(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that the absolute values $|\tilde{f}(w^\ell) - w^\ell| < \epsilon$ for all $w^\ell \in X^\ell$. Note that if all utility functions u^ℓ arise from such perturbations, we obtain in particular

$$\sum_{\ell=1}^{L} (u^\ell(\theta^\ell) - \theta^\ell) > \sum_{\ell=1}^{L} (u^\ell(w^\ell) - w^\ell) \tag{B.6}$$

for all $w \in X \setminus \{\theta\}$ (note that every $w \in X \setminus \{\theta\}$ differs from θ in at least one coordinate, hence the inequality in (B.6) is indeed strict). Since $\sum_{\ell=1}^{L} \theta^\ell = \sum_{\ell=1}^{L} w^\ell = Q$, this implies $\sum_{\ell=1}^{L} u^\ell(\theta^\ell) > \sum_{\ell=1}^{L} u^\ell(w^\ell)$ for all $w \in X \setminus \{\theta\}$, i.e. θ is the unique top of the preference ordering represented by the utility function $u = \sum_\ell u^\ell$.

Now let $x \notin [\theta, y]$ and assume with loss of generality that x, y, θ are pairwise distinct. Since $x \notin [\theta, y]$ there exists a coordinate $j = 1, ..., L$ such that $x^j \notin [\theta^j, y^j]$. Thus, either $(x^j < \theta^j \;\&\; x^j < y^j)$ or $(x^j > \theta^j \;\&\; x^j > y^j)$. Consider the first case. It is possible to choose, for any position of θ^j and y^j, a strictly increasing and strictly concave function $u^j : X^l \to \mathbb{R}$ such that

$$u^j(\theta^j) - \theta^j \geq u^j(y^j) - y^j \geq \delta > 0 \geq u^j(x^j) - x^j, \tag{B.7}$$

where the first inequality in (B.7) is strict whenever $\theta^j \neq y^j$, and such that the difference $u^j(\theta^j) - \theta^j$ is in fact strictly larger than $u^j(w^j) - w^j$ for all $w^j \in X^j \setminus \{\theta^j\}$. Figure 16 depicts the two cases $\theta^j < y^j$ (left) and $y^j < \theta^j$ (right).

Now choose all other functions u^ℓ strictly increasing and strictly concave such that $u^\ell(\theta^\ell) - \theta^\ell$ is strictly larger than $u^\ell(w^\ell) - w^\ell$ for all $w^\ell \in X^\ell \setminus \{\theta^\ell\}$, and such that $|u^\ell(w^\ell) - w^\ell| < \delta/[2(L-1)]$ for all $w^\ell \in X^\ell$, as described above. Let $\succcurlyeq$ be the preference order represented by $u = \sum_{\ell=1}^{L} u^\ell$. As argued above, θ is the top alternative of $\succcurlyeq$. Moreover, we have

$$u^j(y^j) - y^j \;+\; \sum_{\ell \neq j} (u^\ell(y^\ell) - y^\ell) \;>\; \delta/2 \;>\; u^j(x^j) - x^j \;+\; \sum_{\ell \neq j} (u^\ell(x^\ell) - x^\ell),$$

i.e. $u(y) > u(x)$, and hence $y \succ x$ as desired. The argument in the case $x^j > \theta^j$ & $x^j > y^j$ is completely symmetric.

Now consider the model $\boldsymbol{\mathcal{M}}_{\text{lin}}$ of all linear preferences on X. Observe first that only corner allocations can be the top of a linear preference order on X. To show that $\boldsymbol{\mathcal{M}}_{\text{lin}}$ also forms a rich separably convex model, we need to show that for every corner allocation $\theta \in X$ and any pair $x, y \in X$ such that $x \notin [\theta, y]$ one can find an element $\succcurlyeq \in \boldsymbol{\mathcal{M}}_{\text{lin}}$ with top θ and $u(y) > u(x)$. This is obvious if $L = 2$, i.e. in the case of a line; thus, assume $L \geq 3$. Without loss of generality assume that $\theta = (Q, 0, ..., 0)$. We distinguish three cases.

Case 1. If $y^1 > x^1$, choose $a^1 = 1$ and all other a^ℓ pairwise distinct such that $0 < a^\ell < \varepsilon$ for all $\ell \neq 1$. If ε is sufficiently small, we obtain

$$\sum_{\ell=1}^{L} a^\ell \cdot y^\ell \ > \ \sum_{\ell=1}^{L} a^\ell \cdot x^\ell \tag{B.8}$$

i.e. $u(y) > u(x)$ for the linear utility function represented by the a^ℓ; moreover, since $a^1 > a^\ell$ for all $\ell \neq 1$, $\theta = (Q, 0, ..., 0)$ is indeed the top of the corresponding linear preference.

Case 2. If $y^1 = x^1$, then there exists $k > 1$ such that $y^k > x^k$. Choose $a^1 = 1$, $a^k = 1 - \varepsilon$, and all other a^ℓ pairwise distinct such that $0 < a^\ell < \varepsilon$ for all $\ell \notin \{1, k\}$. If ε is sufficiently small the coefficients $\{a^1, ..., a^L\}$ represent a linear preference with top $\theta = (Q, 0, ..., 0)$ and $y \succ x$ as desired.

Case 3. Finally, consider the case $y^1 < x^1$. Let $L^- := \{\ell : y^\ell < x^\ell\}$ and $L^+ := \{\ell : y^\ell > x^\ell\}$. By the feasibility of x and y, we have

$$\sum_{\ell \in L^-} (x^\ell - y^\ell) \ = \ \sum_{\ell \in L^+} (y^\ell - x^\ell) \tag{B.9}$$

with each summand in this equation being strictly positive by construction. By the assumption of the present case, we have $1 \in L^-$, and since $x \notin [\theta, y]$ there exists $k \neq 1$ such that $k \in L^-$. This implies by (B.9)

$$(x^1 - y^1) \ < \ \sum_{\ell \in L^+} (y^\ell - x^\ell). \tag{B.10}$$

Now choose $a^1 = 1$, and all other a^ℓ pairwise distinct such that $1 - \varepsilon < a^\ell < 1$ for $\ell \in L^+$ and $0 < a^\ell < \varepsilon$ for $\ell \in L \setminus (\{1\} \cup L^+)$. For ε sufficiently small, we obtain by (B.10) that $y \succ x$ for the linear preference represented by the coefficients $\{a^1, ..., a^L\}$ as in (B.8). Furthermore, since a^1 is the uniquely largest coefficient, $\theta = (Q, 0, ..., 0)$ is indeed the top of this preference. □

Proof of Lemma 4.1. That $x \in [\theta, y]$ implies $x >_\theta^{\text{sepco}} y$ has been shown in the main text. To see the converse implication, assume by way of contraposition that $x \notin [\theta, y]$. As in the proof of Fact B.2 above, one can construct a separably convex (in fact: additively separable) preference order with top θ such that $y \succ x$, i.e. $x \not>_\theta^{\text{sepco}} y$. This shows (4.6) and completes the proof of Lemma 4.1. □

Proof of Proposition 6. (i) Every finite profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ with support in an ordered domain D can arranged such that, for all $j \leq k \leq l \leq m$, $[\theta_k, \theta_l] \subseteq [\theta_j, \theta_m]$. It follows from

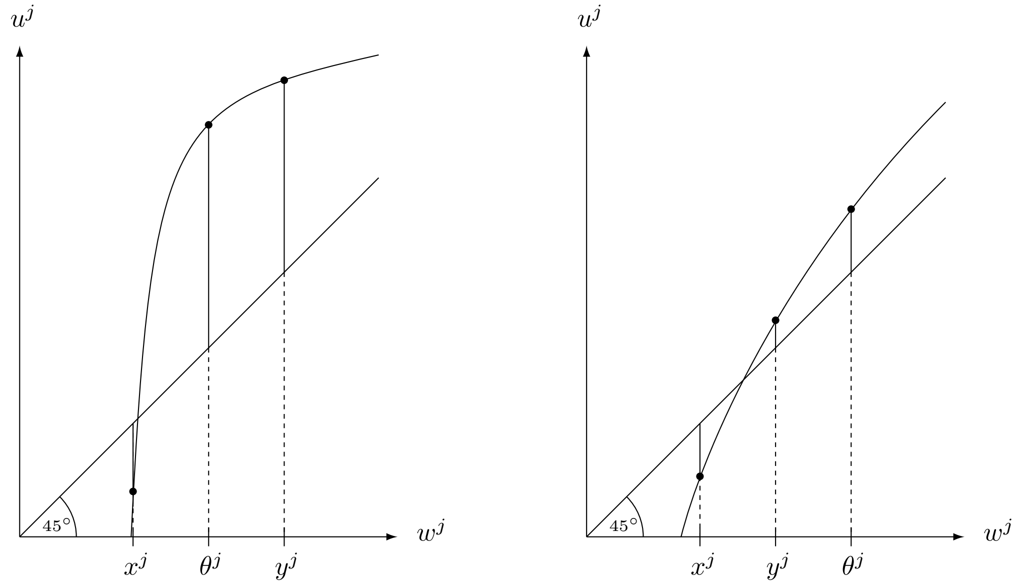

Figure 16: Construction of u^j if $x^j < \theta^j$ and $x^j < y^j$

(4.5) and Lemma 4.1 that $C_{1-\text{med}}(\boldsymbol{\theta}) = \{\theta_{(n+1)/2}\}$ if n is odd and $C_{1-\text{med}}(\boldsymbol{\theta}) = [\theta_{n/2}, \theta_{n/2+1}]$ if n is even. This implies the desired inclusion; indeed, all elements in $\{\theta_1, ..., \theta_n\} \setminus C_{1-\text{med}}(\boldsymbol{\theta})$ are not majority admissible because they are dominated by the median top(s).

(ii) Suppose that $\{\theta_1, \theta_2, \theta_3\}$ are not collinear (as in Fig. 4). Suppose that in a sufficiently large population, 1/2 of the voters have their top at θ_1 while 1/4 of the voters have their top at θ_2 and θ_3, respectively. Then, all p-medians for $p > 1$ uniquely select θ_1 as the solution. This follows from the fact that any movement away from θ_1 by one unit along a geodesic increases the total L_p-distance by 1/2 but reduces it by less than 1/2 because one cannot approach θ_2 and θ_3 on a common geodesic. (Observe that, by contrast, the 1-median chooses all elements on the line segment joining θ_1 and θ_2, precisely because this line segment is on a common geodesic connecting θ_1 with θ_2 and θ_3.) By continuity, θ_1 remains the unique solution if a sufficiently small positive mass is taken from θ_1 and distributed, say, equally, to θ_2 and θ_3.

(iii) First, it is well-known and easily verified that all p–medians coincide with the standard median on a line for $p \geq 1$. Moreover, the median top $\theta_{\text{med}} = \theta_{(n+1)/2}$ is clearly majority admissible. To see that it is the only majority admissible alternative, consider any other allocation $x \neq \theta_{\text{med}}$ and any of its neighbors y in $[x, \theta_{\text{med}}]$. As is easily verified, we have $y \in [x, \theta_i]$ either for all $i \in \{1, ..., \frac{n+1}{2}\}$, or for all $i \in \{\frac{n+1}{2}, ..., n\}$; hence, x is not majority admissible. □

Proof of Fact 4.1. Evidently, since $>_\theta^{\text{sepco}}$ is transitive, it contains the transitive closure of its restriction to pairs of Γ_{res}-neighbors. Conversely, suppose that $x >_\theta^{\text{sepco}} y$ and consider any shortest Γ_{res}-path from θ to y that contains x. Let $w_1, ..., w_m$ be consecutive neighbors on that path with $x = w_1$ and $y = w_m$. As in the proof of Lemma 4.1 in the main text, we obtain $w_j \succ w_{j+1}$ for all $j = 1, ..., m-1$ and all separably convex preference orderings with top θ. In particular, $w_j >_\theta^{\text{sepco}} w_{j+1}$ for all $j = 1, ..., m-1$, i.e. the transitive closure of the restriction of $>_\theta^{\text{sepco}}$ to all neighbors contains the pair (x, y). $\square$

Proof of Fact 4.2. If y is a neighbor of x there is nothing to show. Otherwise, consider any neighbor y' of x in $[y, x]$; by Lemma 4.1, we have $y' >_\theta^{\text{sepco}} x$ for all tops θ such that $y >_\theta^{\text{sepco}} x$. Thus, if y is an absolute majority winner against x, so must be y'. $\square$

The following result provides a key technical tool for showing that any local optimum of a function is in fact a global optimum.

Lemma B.9. *Let $f : X \to \mathbb{R}$ be a separable function with $f(x) = \sum_{\ell=1}^{L} f^\ell(x^\ell)$ such that all functions $f^\ell(\cdot)$ are concave. Then, any local optimum of f on X is also a global optimum of f on X, i.e. if $f(x) \geq f(w)$ for all neighbors $w \in X$ of x, then $f(x) \geq f(w)$ for all $w \in X$. Moreover, the set of optima is box-convex, i.e. every point on a shortest L_1-path between two optima is also an optimum.*

Proof. As in the first part of the proof of Fact B.2, the stated conditions imply that f represents a separably convex preference order on X (recall that no monotonicity condition on f or the f^ℓ is required for this conclusion). This implies that f must be constant along any shortest path connecting two local optima. Indeed, suppose by way of contradiction that f is not constant along some shortest path connecting two local optima x and z. Then there exist two neighbors along that path, say y and $y_{(kj)}$, such that $f(y) < f(y_{(kj)})$. Since y and $y_{(kj)}$ are on a shortest path connecting x and z, we have $y_{(kj)} \in [y, z]$ or $y_{(kj)} \in [y, x]$. Without loss of generality, assume the former; then, by the separable convexity, we obtain $f(z) < f(y_{(kj)})$ contradicting the assumption that z is a local optimum. From this, all other assertions in Lemma B.9 follow at once. $\square$

Proof of Theorem 3. By Lemma 4.2, we obtain for any profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and any two neighbors x and y,

$$\#\{i : x >_{\theta_i}^{\text{sepco}} y\} - \#\{i : y >_{\theta_i}^{\text{sepco}} x\} \; = \; \Delta_1(x; \boldsymbol{\theta}) - \Delta_1(y; \boldsymbol{\theta}).$$

In particular, $xR^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}y$ if and only if $\Delta_1(x; \boldsymbol{\theta}) \leq \Delta_1(y; \boldsymbol{\theta})$. This implies the acyclicity of the local net majority tournament and the inclusion $C_{1-\text{med}}(\boldsymbol{\theta}) \subseteq \text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})$. Note moreover, that a neighbor y of a local ex-ante Condorcet winner x is itself a local ex-ante Condorcet winner if and only if $xR^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}y$ and $yR^{\text{loc}}_{(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})}x$.

For each top θ_i, the negative L_1-distance $-d(x, \theta_i) = \sum_\ell -|x^\ell - \theta_i^\ell|$ is the sum of the concave functions $-|x^\ell - \theta_i^\ell|$, i.e. separable and concave. Hence, as a sum of such functions the negative of aggregate distance $-\Delta_1(\cdot; \boldsymbol{\theta})$ is likewise separable and concave function. By Lemma B.9, each of its local optima is a global optimum. This implies $\text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) \subseteq C_{1-\text{med}}(\boldsymbol{\theta})$, hence by the first part of this proof in fact $\text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}}) = C_{1-\text{med}}(\boldsymbol{\theta})$ for all profiles $\boldsymbol{\theta}$. From this the box-convexity of $\text{CW}^{\text{loc}}(\boldsymbol{\theta}, \boldsymbol{\mathcal{M}}_{\text{sepco}})$ follows using Lemma B.9 again. $\square$

Proof of Proposition 7. The statement follows easily from Theorem 4 below. $\square$

Proof of Theorem 4. The idea of the proof is to show that (4.11) is equivalent to x being a local maximum of aggregate goal satisfaction as defined in Appendix A.5, and then to apply Lemma B.9 and Proposition 12.

We first introduce some notation. For a fixed profile $\boldsymbol{\theta} = (\theta_1, ..., \theta_n) \in X^n$, each $\ell = 1, ..., L$ and $r \in \mathbb{Z}$, denote by $v^\ell(r) := \sum_{i=1}^n \min\{r, \theta_i^\ell\}$ so that for the aggregate goal satisfaction function $v(\cdot)$ we have $v(x) = \sum_\ell v^\ell(x^\ell)$. Moreover, let

$$\begin{array}{rcl}\nabla_- v^\ell(r) & := & v^\ell(r) - v^\ell(r-1), \\ \nabla_+ v^\ell(r) & := & v^\ell(r+1) - v^\ell(r).\end{array}$$

By construction, we obtain

$$\begin{array}{rcl}\nabla_- v^\ell(r) & = & \#\{i : \theta_i^\ell \geq r\}, \\ \nabla_+ v^\ell(r) & = & \#\{i : \theta_i^\ell \geq r+1\}.\end{array} \tag{B.11}$$

By definition of $\theta_{[k]}^\ell$, we have $\#\{i : \theta_i^\ell \geq r\} \geq (n-k+1)$ whenever $r \leq \theta_{[k]}^\ell$, and hence by (B.11),

$$r \leq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_- v^\ell(r) \geq (n-k+1). \tag{B.12}$$

Similarly, we have $\#\{i : \theta_i^\ell \geq r+1\} \leq (n-k)$ whenever $r \geq \theta_{[k]}^\ell$, hence, again by (B.11),

$$r \geq \theta_{[k]}^\ell \;\Rightarrow\; \nabla_+ v^\ell(r) \leq (n-k). \tag{B.13}$$

Now consider any $x \in X$ satisfying (4.11), i.e. for all $\ell = 1, ..., L$, $\theta_{[k^*(\boldsymbol{\theta})]}^\ell \leq x^\ell \leq \theta_{[k^*(\boldsymbol{\theta})+1]}^\ell$. We will show that x is a local maximizer of aggregate goal satisfaction v. By Lemma B.9, x is then also a global optimum, hence a frugal majority winner by Proposition 12. Thus, consider any neighbor y of x. Without loss of generality, assume that $y = x_{(21)}$, i.e. $y^1 = x^1 - 1$, $y^2 = x^2 + 1$, and $y^\ell = x^\ell$ for all $\ell = 3, ..., L$. We have $x^1 \leq \theta_{[k^*(\boldsymbol{\theta})+1]}^\ell$ and $x^2 \geq \theta_{[k^*(\boldsymbol{\theta})]}^\ell$, therefore, using (B.12) and (B.13),

$$\begin{array}{rcl}v(x) - v(y) & = & \nabla_- v^1(x^1) - \nabla_+ v^2(x^2) \\ & \geq & n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - k^*(\boldsymbol{\theta})) \\ & = & 0.\end{array}$$

This proves that every $x \in X$ satisfying (4.11) is indeed a maximizer of aggregate goal satisfaction.

Conversely, consider $x \in X$ that violates (4.11). There are two (not mutually exclusive) cases.

Case 1. For some coordinate h, $x^h < \theta_{[k^*(\boldsymbol{\theta})]}^h$. In this case, there must exist some other coordinate j such that $x^j > \theta_{[k^*(\boldsymbol{\theta})]}^j$. Consider the neighbor y of x such that $y^h = x^h + 1$, $y^j = x^j - 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(hj)}$. By the same arguments as above, we obtain using (B.11),

$$r < \theta_{[k]}^\ell \;\Rightarrow\; \nabla_+ v^\ell(r) \geq (n-k+1) \tag{B.14}$$

and

$$r > \theta^\ell_{[k]} \;\Rightarrow\; \nabla_- v^\ell(r) \leq (n-k). \tag{B.15}$$

Therefore,

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^h(x^h) - \nabla_- v^j(x^j) \\ &\geq n - k^*(\boldsymbol{\theta}) + 1 - (n - k^*(\boldsymbol{\theta})) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction.

Case 2. For some coordinate h, $x^h > \theta^h_{[k^*(\boldsymbol{\theta})+1]}$. In this case, there must exist some other coordinate ℓ such that $x^\ell < \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}$. Consider the neighbor y of x such that $y^h = x^h - 1$, $y^\ell = x^\ell + 1$, and $y^\ell = x^\ell$ for all coordinates $\ell \neq h, \ell$, i.e. $y = x_{(jh)}$. By (B.14) and (B.15), we obtain

$$\begin{aligned} v(y) - v(x) &= \nabla_+ v^j(x^j) - \nabla_- v^h(x^h) \\ &\geq n - (k^*(\boldsymbol{\theta}) + 1) + 1 - (n - (k^*(\boldsymbol{\theta}) + 1)) \\ &= 1, \end{aligned}$$

hence x is not a maximizer of aggregate goal satisfaction in this case either. This completes the proof of Theorem 4. □

Proof of Proposition 8. Follows at once from Theorem 4. □

For the proof of Proposition 9, we need the following lemma.

Lemma B.10. *Let $x \in C_{1-\mathrm{med}}(\boldsymbol{\theta})$. Then,*

$$\widetilde{\mathfrak{d}}{\downarrow}(x;\boldsymbol{\theta}) = k^*(\boldsymbol{\theta}). \tag{B.16}$$

If $C_{1-\mathrm{med}}(\boldsymbol{\theta})$ is a singleton, i.e. $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{x\}$, then $\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}) \geq n - k^*(\boldsymbol{\theta}) + 1$, where n is the number of voters; otherwise, if $C_{1-\mathrm{med}}(\boldsymbol{\theta})$ is not a singleton, then

$$\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}) = n - k^*(\boldsymbol{\theta}). \tag{B.17}$$

Proof. By Theorem 4, we have

$$C_{1-\mathrm{med}}(\boldsymbol{\theta}) \;=\; \left\{ x \in X \mid \theta^\ell_{[k^*(\boldsymbol{\theta})]} \leq x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]} \text{ for all } \ell = 1, ..., L \right\},$$

where $\theta^\ell_{[k]}$ is the k-th lowest value in the set $\{\theta^\ell_i : i = 1, ..., n\}$, and $k^*(\boldsymbol{\theta})$ the largest k such that $\sum_{\ell=1}^L \theta^\ell_{[k]} \leq Q$. Therefore, equation (B.16) holds by definition. If $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{x\}$, then $\theta^\ell_{[k^*(\boldsymbol{\theta})]} = x^\ell$ for all ℓ, hence $\boldsymbol{\theta}(H^\ell_{x\uparrow}) \geq n - (k^*(\boldsymbol{\theta}) - 1) = n - k^*(\boldsymbol{\theta}) + 1$ for all ℓ; thus, $\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}) \geq n - k^*(\boldsymbol{\theta}) + 1$.[19]

[19] Observe that $\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta})$ can be much larger than $n - k^*(\boldsymbol{\theta})$; for instance, if $\boldsymbol{\theta}$ is an unanimous profile, we have $\widetilde{\mathfrak{d}}{\downarrow}(x;\boldsymbol{\theta}) = k^*(\boldsymbol{\theta}) = n$ as well as $\widetilde{\mathfrak{d}}{\uparrow}(x;\boldsymbol{\theta}) = n$.

Finally, suppose that $x, y \in C_{1-\mathrm{med}}(\boldsymbol{\theta})$ for distinct $x, y \in X$. For all ℓ, we have $x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}$ and thus $\boldsymbol{\theta}(H^\ell_{x\uparrow}) \geq n - k^*(\boldsymbol{\theta})$; this implies $\widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta}) \geq n - k^*(\boldsymbol{\theta})$. Moreover, for some ℓ_0, $x^{\ell_0} > y^{\ell_0}$; since $y^{\ell_0} \geq \theta^{\ell_0}_{[k^*(\boldsymbol{\theta})]}$, this implies $x^{\ell_0} > \theta^{\ell_0}_{[k^*(\boldsymbol{\theta})]}$, and hence $\boldsymbol{\theta}(H^\ell_{x\uparrow}) = n - k^*(\boldsymbol{\theta})$. Thus, $\widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta}) = n - k^*(\boldsymbol{\theta})$ as asserted in (B.16). □

Proof of Proposition 9. It will be convenient to introduce the following notation. For all profiles $\boldsymbol{\theta}$, denote

$$\begin{aligned} C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) &:= \arg\max_{x \in X} \widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta}), \\ C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta}) &:= \arg\max_{x \in X} \widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta}). \end{aligned}$$

We will show that, for all $\boldsymbol{\theta}$, $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) \cap C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta})$, i.e. $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \widetilde{T}^*(\boldsymbol{\theta})$. In addition, if $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{x\}$ for some $x \in X$, then $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) = C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta})$. Consider first the case $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{x\}$ for some $x \in X$. As noted in the proof of Lemma B.10, one then has $\theta^\ell_{[k^*(\boldsymbol{\theta})]} = x^\ell$ for all ℓ; moreover, $x \in C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta})$ and $x \in C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta})$. If $y \neq x$, then $y^k < x^k$ and $y^m > x^m$ for some $k, m = 1, ..., L$, hence $\boldsymbol{\theta}(H^k_{y\downarrow}) < \boldsymbol{\theta}(H^k_{x\downarrow})$ and $\boldsymbol{\theta}(H^m_{y\uparrow}) < \boldsymbol{\theta}(H^m_{x\uparrow})$. Therefore, $\widetilde{\mathfrak{d}}\uparrow(y;\boldsymbol{\theta}) < \widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta})$ and $\widetilde{\mathfrak{d}}\downarrow(y;\boldsymbol{\theta}) < \widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta})$, hence $C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) = C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta}) = \{x\}$.

Now consider the case in which $C_{1-\mathrm{med}}(\boldsymbol{\theta})$ is not a singleton. As in the proof of Lemma B.10 we have $\arg\max_{x \in X} \widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta}) = k^*(\boldsymbol{\theta})$ and $\arg\max_{x \in X} \widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta}) = n - k^*(\boldsymbol{\theta})$, hence

$$C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta}) = \{x \in X \mid \theta^\ell_{[k^*(\boldsymbol{\theta})]} \leq x^\ell \text{ for all } \ell\}.$$

and

$$C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) = \{x \in X \mid x^\ell \leq \theta^\ell_{[k^*(\boldsymbol{\theta})+1]} \text{ for all } \ell\}.$$

By Theorem 4, we have $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta}) \cap C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta}) = \widetilde{T}^*(\boldsymbol{\theta})$. Note in particular, that the intersection of $C_{\mathrm{chs}\uparrow}(\boldsymbol{\theta})$ and $C_{\mathrm{chs}\downarrow}(\boldsymbol{\theta})$ is thus always non-empty. Moreover, we evidently have $C_{1-\mathrm{med}}(\boldsymbol{\theta}) \subseteq \widetilde{T}(\boldsymbol{\theta})$. □

The inclusion $\widetilde{T}^*(\boldsymbol{\theta}) \subseteq \widetilde{T}(\boldsymbol{\theta})$ is in general strict as shown by the following example.

Example 9. *Consider $\boldsymbol{\theta}$ with three voters at $\theta_1 = (2, 0, 0)$, $\theta_2 = (0, 1, 1)$ and $\theta_3 = (0, 0, 2)$. For the point $x = (1, 1, 0)$ we have $\widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta}) = \widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta}) = 1$, while for every allocation $w \in C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{(2, 0, 0), (0, 1, 1), (1, 0, 1)\}$ we have $\widetilde{\mathfrak{d}}\uparrow(w;\boldsymbol{\theta}) = 2$ and $\widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta}) = 1$. Hence, $x \in \widetilde{T}(\boldsymbol{\theta}) \setminus \widetilde{T}^*(\boldsymbol{\theta})$.*

Also observe that there is no refinement in the case in which $C_{1-\mathrm{med}}(\boldsymbol{\theta}) = \{x\}$, because in that case $\boldsymbol{\theta}(H^\ell_{x\downarrow}) = \boldsymbol{\theta}(H^{\ell'}_{x\downarrow})$ and $\boldsymbol{\theta}(H^\ell_{x\uparrow}) = \boldsymbol{\theta}(H^{\ell'}_{x\uparrow})$ for all ℓ, ℓ'. Nevertheless, one might well have $\widetilde{\mathfrak{d}}\downarrow(x;\boldsymbol{\theta}) \neq \widetilde{\mathfrak{d}}\uparrow(x;\boldsymbol{\theta})$.

Proof of Proposition 10. The statement follows in a straightforward manner from Theorem 4. Indeed, order all projects in terms of their individual approvals. By Theorem 4, the ex-ante Condorcet solution must fund the most popular projects until the budget is exhausted. With the residual money the next popular project must be funded with some probability. Note that there could be several equally popular projects, therefore the Condorcet solution might contain allocations in which more than one project is funded with non-trivial probability; but there is always also one allocation among the Condorcet solutions in which all the residual money goes to one project only. □

Proof of Proposition 13. The idea of the proof is to use Theorem 4 in order to show that by participating an agent moves the interval $[\theta^\ell_{[k^*(\cdot)]}, \theta^\ell_{[k^*(\cdot)+1]}]$ 'closer' to her top in all coordinates ℓ simultaneously.

The statement of Proposition 13 is easily verified for a unanimous profile, thus assume in the following that $\boldsymbol{\theta}$ is non-unanimous. Then there exists $k^*(\boldsymbol{\theta}) < n$ such that $Q_{[k^*(\boldsymbol{\theta})]} \leq Q$ and $Q_{[k^*(\boldsymbol{\theta})+1]} > Q$ as in Theorem 4. Now consider the additional participation of agent h with top θ_h. We use the following notation: the profile $\boldsymbol{\theta} \sqcup \theta_h$ will also be denoted $\tilde{\boldsymbol{\theta}}$; for each ℓ, $\tilde{\theta}^\ell_{[1]} \leq ... \leq \tilde{\theta}^\ell_{[n+1]}$ are the $n+1$ ordered values among $\{\theta^\ell_1, ..., \theta^\ell_n, \theta^\ell_h\}$, and

$$\tilde{Q}_{[k]} := \sum_{\ell=1}^{L} \tilde{\theta}^\ell_{[k]}$$

Since, for each ℓ, both the values $\theta^\ell_{[k]}$ and $\tilde{\theta}^\ell_{[k]}$ are weakly increasing in k, we obtain, for all $k \leq n$,

$$\tilde{Q}_{[k]} \leq Q_{[k]}.$$

Moreover, by the addition of agent h, we have $\theta^\ell_{[k-1]} \leq \tilde{\theta}^\ell_{[k]}$ for all ℓ and $k \leq n$, and hence

$$\tilde{Q}_{[k+1]} \geq Q_{[k]}.$$

In particular, we obtain $\tilde{Q}_{[k^*(\boldsymbol{\theta})]} \leq Q_{[k^*(\boldsymbol{\theta})]} \leq Q$, and $\tilde{Q}_{[k^*(\boldsymbol{\theta})+2]} \geq Q_{[k^*(\boldsymbol{\theta})+1]} > Q$. Thus, there are only two cases, either (i) $k^*(\tilde{\boldsymbol{\theta}}) = k^*(\boldsymbol{\theta})$, or (ii) $k^*(\tilde{\boldsymbol{\theta}}) = k^*(\boldsymbol{\theta}) + 1$.

In either case, it follows immediately from the definitions that, for all ℓ, the interval $[\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})]}, \tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})+1]}]$ is 'closer' to θ^ℓ_h than the interval $[\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_{[k^*(\boldsymbol{\theta})+1]}]$ in the sense that both

$$\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})]} \in \left[\theta^\ell_{[k^*(\boldsymbol{\theta})]}, \theta^\ell_h\right],$$

and

$$\tilde{\theta}^\ell_{[k^*(\tilde{\boldsymbol{\theta}})+1]} \in \left[\theta^\ell_{[k^*(\boldsymbol{\theta})+1]}, \theta^\ell_h\right].$$

This implies the two inclusions stated in (A.4) and completes the proof of Proposition 13. □

Proof of Theorem 3′. Lemma 4.1 continues to hold in the continuous case and we still have $x >^{\text{sepco}}_\theta y \Leftrightarrow x \in [\theta, y]$. As in the proof of Theorem 3 above, we also have, for all profiles $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and every pair x, y with $x \Gamma_{\boldsymbol{\theta}} y$,

$$\#\{i : x >^{\text{sepco}}_{\theta_i} y\} - \#\{i : y >^{\text{sepco}}_{\theta_i} x\} \;=\; \Delta_1(x; \boldsymbol{\theta}) - \Delta_1(y; \boldsymbol{\theta}). \tag{B.18}$$

In the continuous case, (B.18) can be derived as follows. Assume without loss of generality that $x^j > y^j$, $x^k < y^k$, $x^\ell = y^\ell$ for all $\ell \neq j, k$, and consider any top θ_i. By condition (ii) in the definition of $\Gamma_{\boldsymbol{\theta}}$, there are four cases (corresponding to the four non-shaded regions in Fig. 15 above):

(a) $\theta_i^j \geq x^j$ and $\theta_i^k \leq x^k$,

(b) $\theta_i^j \geq x^j$ and $\theta_i^k \geq y^k$,

(c) $\theta_i^j \leq y^j$ and $\theta_i^k \leq x^k$, or

(d) $\theta_i^j \leq y^j$ and $\theta_i^k \geq y^k$.

In case (a), we have $x \in [\theta_i, y]$ and hence $d(\theta_i, y) = d(\theta_i, x) + d(x, y)$; in case (d) we have $y \in [\theta_i, x]$ and hence $d(\theta_i, x) = d(\theta_i, y) + d(x, y)$. In cases (b) and (c), we have neither $x \in [\theta_i, y]$ nor $y \in [\theta_i, x]$, and since $|x^j - y^j| = |x^k - y^k|$ by the feasibility of x and y, we obtain $d(\theta_i, x) = d(\theta_i, y)$ in either of these two cases. Thus, for all supporters of x over y the distance of their top to y is by $d(x, y)$ larger than the distance of their top to x; for all supporters of y over x the distance of their top to x is by $d(x, y)$ larger than the distance of their top to y; and for all other the distance of their top to x is the same as the distance of their top to y; this implies (B.18).

The rest of the proof follows from straightforward adaption of the arguments given in the proof of Theorem 3. In particular, Lemma B.9 generalizes in a straightforward manner to the continuous case. $\square$

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