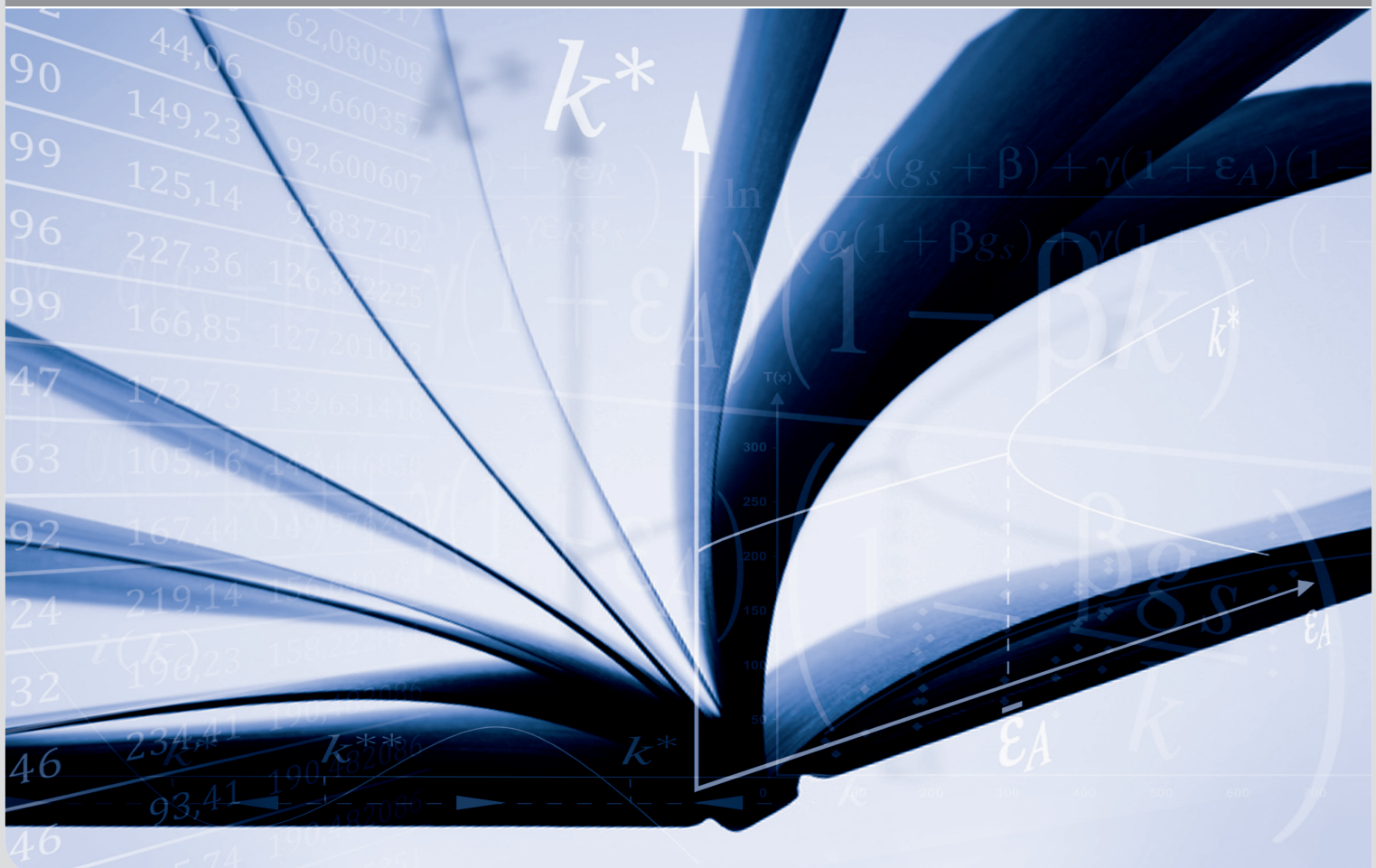


A note on Condorcet consistency and the median voter

by Berno Buechel

No. 17 | APRIL 2011

WORKING PAPER SERIES IN ECONOMICS



Impressum

Karlsruher Institut für Technologie (KIT)
Fakultät für Wirtschaftswissenschaften
Institut für Wirtschaftspolitik und Wirtschaftsforschung (IWW)
Institut für Wirtschaftstheorie und Statistik (ETS)

Schlossbezirk 12
76131 Karlsruhe

KIT – Universität des Landes Baden-Württemberg und
nationales Forschungszentrum in der Helmholtz-Gemeinschaft

Working Paper Series in Economics
No. 17, April 2011

ISSN 2190-9806

econpapers.wiwi.kit.edu

A Note on Condorcet Consistency and the Median Voter

Berno Buechel (berno@wiwi.uni-saarland.de)
Chair of Economic Theory - Saarland University

April 5, 2011

Abstract

We discuss to which extent the median voter theorem extends to the domain of single-peaked preferences on median spaces. After observing that on this domain a Condorcet winner need not exist, we show that if a Condorcet winner does exist, then it coincides with the median alternative (“the median voter”). Based on this result, we propose two non-cooperative games that implement the unique strategy-proof social choice rule on this domain.

1 Introduction

The median voter theorem (henceforth: MVT) is one of the central theorems in economic theory. It applies to agents who have to choose one element out of a set of alternatives. In its classic formulation, the MVT consists of the following two statements (while in many instances only one of them is mentioned):

If preferences are single-peaked (on a line), then

- (i) there is a neutral and anonymous social choice rule that is both non-dictatorial and strategy-proof. The unique rule of this type is to select the favorite alternative of the median voter (Mas-Colell, Whinston, and Green 1995, e.g.).
- (ii) The median voter’s favorite alternative wins in majority voting against any other alternative, i.e. it is a Condorcet winner (Congleton 2002, e.g.).

While the first part shows that one can “escape” the negative result of the Gibbard-Satterthwaite-Theorem, the second part shows that Condorcet’s paradox can not occur. The second part is strongly related to a Hotelling-Downs model of political competition: In a game between two

vote maximizing candidates, the unique equilibrium strategy is to choose the Condorcet winner.

In a seminal contribution Nehring and Puppe (2007b) show that the MVT part (i) can be extended to a much larger class of preferences—single-peaked preferences on median spaces, which contain lines, trees, grids, and hypercubes as special cases. As a starting point of this paper we provide a simple example showing that this extension of the MVT need not hold for part (ii) of the MVT, i.e. the median alternative need not be a Condorcet winner. However, we show for this domain that the median alternative is the only candidate for a Condorcet winner (Prop. 1), i.e. the corresponding social choice function is *Condorcet consistent*. Thus, if there is a Nash equilibrium between two opportunistic (vote maximizing) candidates, then it must be that both candidates choose the median position (Cor. 1). Moreover, we show that this result is not restricted to opportunistic candidates. Assuming that politicians are reformists, i.e. they care about the winning policy (rather than being the winner), this result still holds if the preferences of the two candidates are “sufficiently heterogeneous” (Prop. 2). Those results have important consequences for the implementation of the unique strategy-proof social choice rule, which we discuss to conclude this note.

2 Set-up

Let $X = \{1, 2, \dots\}$ be a finite (and fixed) set of *alternatives* (e.g. social states, policies, political positions) of size $|X| \geq 3$. Let $N = \{1, 2, \dots, n\}$ be a finite set of *voters*. The voters are endowed with (complete and transitive) preferences on the alternatives X . $\succeq := (\succeq_i)_{i \in N}$ denotes a profile of such preferences and \mathcal{D} stands for the domain of all such preferences. A Condorcet winner is an alternative x satisfying the following property: $\forall y \in X$, it holds that $\#\{i \in N \mid x \succ_i y\} \geq \#\{i \in N \mid y \succ_i x\}$.

In order to define the relevant domain of preferences, we define property spaces and generalized single-peakedness following Nehring and Puppe (2007b). A set of *basic (binary) properties* \mathcal{H} is extensionally defined via the alternatives: $\mathcal{H} \subseteq 2^X$, where $H \in \mathcal{H}$ stands for a property possessed by exactly all alternatives $x \in H$. A pair (X, \mathcal{H}) is called a *property space* if for each property $H \in \mathcal{H}$ it holds that it is non-empty and $H^c \in \mathcal{H}$; and for each pair of alternatives $x \neq y$ there is a property such that $x \in H$ and $y \notin H$. A pair (H, H^c) is an *issue*.

A natural relation for a property space is to say that y is *between* x and z if it shares all of their common properties.

Definition 1 (Betweenness) *Let (X, \mathcal{H}) be a property space. $T_{\mathcal{H}}(\subset X \times X \times X): \forall x, y, z \in X$*

$$(x, y, z) \in T_{\mathcal{H}} \iff [\forall H \in \mathcal{H} : \{x, z\} \subseteq H \implies y \in H]. \quad (1)$$

By definition $(x, x, y) \in T_{\mathcal{H}}$ for any x, y , and \mathcal{H} . The class of property spaces under consideration satisfies the following condition: A property space (X, \mathcal{H}) is a *median space* if $\forall x, y, z \in X$, there exists a m such that

$$\{(x, m, y), (x, m, z), (y, m, z)\} \subseteq T_{\mathcal{H}}.$$

Median spaces have several desirable properties (Van de Vel 1993), some of which we are going to exploit.¹ Let a *segment* be the set of alternatives between two alternatives: $[x, y] := \{w \in X : (x, w, y) \in T_{\mathcal{H}}\}$. Two alternatives $x \neq y$ are *neighbors* if $[x, y] = \{x, y\}$. Let $N(x)$ denote the set of neighbors of alternative x .

The central assumption on preferences is that they are single-peaked with respect to a given median space:

Definition 2 *A profile of preferences \succeq (on X) is single-peaked on the property space \mathcal{H} if any voter's preferences are single-peaked w.r.t. to the betweenness relation $T_{\mathcal{H}}$. That is: for each $i \in N$, there exists $x_i^* \in X$ such that*

$$\forall y \neq z \in X, (x_i^*, y, z) \in T_{\mathcal{H}} \implies y \succ_i z. \quad (2)$$

For a characterization and an excellent discussion of single-peaked preferences (on median spaces) we refer the reader, again, to Nehring and Puppe (2007b).

3 The generalization of the median voter theorem

For a profile \succeq of single-peaked preferences and a property $H \in \mathcal{H}$, let $w(H) = \#\{i \in N \mid x_i^* \in H\}$. The set of *median alternatives* is defined as $M(X, \mathcal{H}, w) := \bigcap_{H \in \mathcal{H}: w(H) \geq \frac{n}{2}}$, i.e. the alternatives that only possess majority properties.² If the profile of preferences is a median space, then a median alternative always exists (Nehring and Puppe 2007b). To ease the exposition, let us assume that the property space is non-degenerate, i.e. there is no property H such that $w(H) = \frac{n}{2} = w(H^c)$, e.g. by assuming that n is odd. Then the median alternative is uniquely determined. Fixing some X , let $\mathcal{M} \subset \mathcal{D}$ be the set of preferences profiles, for which there exists a non-degenerate median space \mathcal{H} such that preferences are single-peaked on \mathcal{H} ; and let $f : \mathcal{M} \rightarrow X$ be the rule that selects the median alternative, i.e. $f(\succeq) = M(X, \mathcal{H}, w)$.

¹In particular, for any median space there is a graph (X, G) that represents it in the sense that $(x, y, z) \in T_{\mathcal{H}}$ if and only if y lies on a shortest path between x and z .

²In a graph corresponding to the median space, the median is the node that minimizes the weighted graph distances, where the weights are the number of voters with each node (alternative) as favorite alternative, see, e.g., Hansen, Thisse, and Wendell (1986).

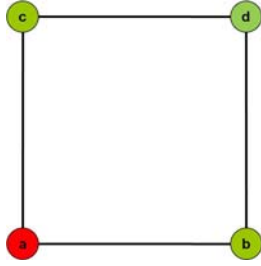


Figure 1: Condorcet’s paradox with single-peaked preferences on a median space.

Nehring and Puppe (2007b) show in their seminal contribution that median spaces are the largest single-peaked domain of preferences to which the MVT extends. (The formulation there is for linear orders and an odd number of voters.)

Theorem 1 (Nehring & Puppe, 2007) *f is neutral, anonymous, non-dictatorial, and strategy-proof.*

This is a substantial generalization of the MVT part (i). Part (ii) does not extend to this large domain as the following example shows.

Example 1 *Let $X = \{a, b, c, d\}$ and $N = \{1, 2, 3\}$ with the following preferences $\succeq_1 = (a, b, c, d)$, $\succeq_2 = (b, d, a, c)$, $\succeq_3 = (c, d, a, b)$. This profile of preferences is single-peaked on the property space $\mathcal{H} = \{Down = \{a, b\}, Up = \{c, d\}, Left = \{a, c\}, Right = \{b, d\}\}$ (see Figure 1). Moreover, this is a median space. The weights are $w(Down) = 2$, $w(Up) = 1$, $w(Left) = 2$, and $w(Right) = 1$. The median alternative is $f(\succeq) = M(X, \mathcal{H}, w) = Left \cap Down = a$. We observe that alternative a is not a Condorcet winner because d would defeat a by two votes over one.³ Note that in this example no Condorcet winner exists. b and c are defeated by a ; and d is defeated, e.g., by b .*

The paradox of Condorcet illustrated in example 1 is well-known for general preferences, but it cannot occur for single-peaked preferences on lines (by the classic MVT part i). The existence of a Condorcet winner can be extended to tree graphs (Demange 1982), i.e. one-dimensional median spaces according to the classification of Nehring and Puppe (2007a). As Example 1 shows, however, the result does not extend, to two-dimensional median spaces.

³By definition a would win a majority voting on the issues, but not majority voting on the alternatives.

Instead of restricting attention to a domain where existence is guaranteed—e.g., single-peaked preferences on one-dimensional median spaces \mathcal{T} —, we address all median spaces and ask whether f is *Condorcet consistent*, i.e. whether f selects the Condorcet winner in case it exists. Let \mathcal{C} designate the set of all preference profiles with a unique Condorcet winner and define $\check{\mathcal{M}} := \mathcal{C} \cap \mathcal{M}$ (i.e. $\check{\mathcal{M}}$ designates the domain of single-peaked preferences on a median space for which a Condorcet winner exists). Then $\mathcal{T} \subset \check{\mathcal{M}} \subset \mathcal{M}$. The following result shows that the median alternative is the only candidate for a Condorcet winner, i.e. f is Condorcet consistent.⁴

Proposition 1 *Suppose $\succeq \in \mathcal{M}$, i.e. \succeq is single-peaked on a median space. If there exists a Condorcet winner, then it is the median alternative $f(\succeq)$.*

Proof. Let $\succeq \in \mathcal{M}$ and $f(\succeq) = q$. Recall that for any $H \in \mathcal{H}$, if $w(H) > \frac{n}{2}$, then $q \in H$. Consider any alternative x with $x \neq q$.

1. $[x, q] \cap N(x) \neq \{\emptyset\}$ because a median space is connected via pairs of neighbors.
2. Let $y \in [x, q] \cap N(x)$. In median spaces neighbors differ in exactly one basic issue (Lemma B.3 in Nehring and Puppe (2007b)). Let (H, H^c) be the issue that separates x from y , with $x \in H$, and $y \in H^c$.
3. $\forall z \in H^c$, we have $(z, y, x) \in T_{\mathcal{H}}$. This is because any property that is shared by z and x is also shared by y (since x and y only differ with respect to (H, H^c)). Thus single-peakedness implies $\forall i \in N$ s.t. $x_i^* \in H^c$, that $y \succ_i x$.
4. $q \in H^c$ because $(q, y, x) \in T_{\mathcal{H}}$ (by definition of y). Thus, $w(H^c) > \frac{n}{2} > w(H)$ (by definition of q). Thus, y defeats x (in the sense that strictly more voters vote for y against x),
5. Thus, any $x (\neq q)$ cannot be a Condorcet winner. If there is a Condorcet winner, then it must be q .

■

The condition $\succeq \in \mathcal{M}$ is necessary for this result. If preferences are single-peaked on a non-median space, then a median alternative need not exist and even if it does, the median alternative and the Condorcet winner do not coincide, in general. Prop. 1 is now used to solve two non-cooperative games of the Hotelling–Downs type.

⁴This is well-known for symmetric single-peaked preferences on trees (Hansen, Thisse, and Wendell 1986). However, the domain here is much larger since we also address multi-dimensional median spaces.

4 Interpretation in terms of political candidates

Consider two political candidates A and B who are able to express a political position. Formally, a strategy for each candidate ($k \in \{A, B\}$) is to pick an alternative $s^k \in S^k = X$. Let the *strategy space* be $S = S^A \times S^B$. Given a strategy profile $s \in S$, each voter's preferences over the alternatives X induce preferences over the candidates A, B . For a given preference profile \succsim , and a strategy profile s , let the *outcome rule* $\phi : \mathcal{D} \times S \rightarrow \mathbb{R}^2$ keep track of how many voters prefer s^A over s^B , where indifferent voters are counted with weight $\frac{1}{2}$ in both entries $\phi^A(\succsim, s)$ and $\phi^B(\succsim, s)$. Let $\rho(\succsim, s)$ be the *winning alternative*, that is the strategy of the candidate with a majority of voters—more precisely, $\rho(\succsim, s) = s^A$ if and only if $\phi^A(\succsim, s) \geq \phi^B(\succsim, s)$.⁵

For the payoffs of the candidates we consider two different assumptions. If the candidates want to maximize their number of votes, $\pi^k(s) := \Pi^k(\phi^k(\succsim, s))$ for some increasing function Π^k , we will call them *opportunists*. Π^k need not be strictly increasing; however, we assume that $\Pi^k(\frac{n}{2} - \varepsilon) < \Pi^k(\frac{n}{2}) < \Pi^k(\frac{n}{2} + \varepsilon)$. Let a *reformist* be a candidate who is not concerned about winning the election, but about which political position is winning the election $\rho(s)$ (because this determines the policy that is finally implemented). So, we here assume that the candidates themselves are endowed with preferences such as voters are—in particular we will assume that the candidates' preferences are single-peaked, as well. Let \succeq^k stand for the preferences of candidate $k = A, B$ on the set X . A reformist k ranks strategy profiles in the following way: s is preferred to s' if and only if $\rho(s) \succeq^k \rho(s')$.

Both assumptions constitute a normal-form game: One for opportunists $\Gamma^\succsim = (\mathcal{K}, S, (\pi^A, \pi^B))$ and one for reformists $\tilde{\Gamma}^\succsim = \{\mathcal{K}, S, (\succeq^A, \succeq^B)\}$. The games are set-up simultaneously, while sequential moves would not change the results. We are interested in whether the median rule f can be implemented by such a game. In a game of opportunists a strategy profile is a Nash equilibrium if and only if it is a Condorcet winner. This leads to the following corollary of Prop. 1.

Corollary 1 (Opportunists) *Let Γ^\succsim be a game of two opportunists with $\succsim \in \mathcal{M}$, i.e. \succsim is single-peaked on a median space. Then*

$$(s^A, s^B) \in NE(\Gamma^\succsim) \implies s^A = s^B = f(\succsim). \quad (3)$$

If an equilibrium exists—that is if $\succsim \in \tilde{\mathcal{M}}$ —, then both candidates choose the median alternative. This includes, for example, single-peaked preferences on one-dimensional median spaces (corresponding to a tree graph).

Now, consider two reformists A, B who have single-peaked preferences on a property space \mathcal{H} with median alternative q . We say that preferences of

⁵If there is a tie, we let A be the winning candidate. If $s^A = s^B$, then this convention does not matter.

two reformists are *sufficiently heterogeneous* if it holds that $(x_*^A, q, x_*^B) \in T_{\mathcal{H}}$. For a one-dimensional median space this means that their favorite alternatives are not in the same branch of the property space, i.e. the same leaf of the corresponding tree (where the leaves are defined with respect to the median alternative q). Note that $(x_*^A, q, x_*^B) \in T_{\mathcal{H}}$ if and only if it holds that $\{x_*^A, x_*^B\} \subseteq H$ implies that $q \in H$, that is x_*^A and x_*^B do not share a property H with $w(H) < \frac{n}{2}$.

Proposition 2 (Reformists) *Suppose $\succeq \in \tilde{\mathcal{M}}$, i.e. \succeq is single-peaked on a median space \mathcal{H} and there exists a Condorcet winner. Let $q = f(\succeq)$ designate the median alternative (and the Condorcet winner). Let $\tilde{\Gamma}^{\succeq}$ be a game of two reformists, where (\succeq^A, \succeq^B) is single-peaked on \mathcal{H} . If the candidates' preferences are sufficiently heterogeneous, i.e. $(x_*^A, q, x_*^B) \in T_{\mathcal{H}}$, then (i) $(q, q) \in NE(\tilde{\Gamma}^{\succeq})$ and (ii) $s^* \in NE(\tilde{\Gamma}^{\succeq}) \Rightarrow \rho(s^*) = q$.*

Proof. Let $\succeq \in \tilde{\mathcal{M}}$, $f(\succeq) = q$, and let $(x_*^A, q, x_*^B) \in T_{\mathcal{H}}$.

(i) Since q is a Condorcet winner (by Prop. 1), $\rho(q, s^k) = q$ for any s^k with $k \in \{A, B\}$. Thus, $\nexists s^{k'}$ that is an improvement for k . Thus, $(q, q) \in NE(\tilde{\Gamma}^{\succeq})$.

(ii) We show that if s is such that $\rho(s) \neq q$, then $s \notin NE(\tilde{\Gamma}^{\succeq})$.

Let $\rho(s) \neq q$, w.l.o.g. let $\rho(s) = s^A$. Let $Y := [x_*^B, s^A] \cap [q, s^A]$. Note first that $s^A \in Y$.

Case 1; $Y = \{s^A\}$. Since \mathcal{H} is a median space, there exists an alternative between the triple q, s^A, x_*^B . Thus, $(q, s^A, x_*^B) \in T_{\mathcal{H}}$. Sufficient heterogeneity means that $(x_*^A, q, x_*^B) \in T_{\mathcal{H}}$. Therefore, $(x_*^A, q, s^A) \in T_{\mathcal{H}}$.

single-peakedness implies that $q \succ^A s^A = \rho(s)$. However, for (any) s^B it holds that $\rho(q, s^B) = q$ because q is a Condorcet winner (by Prop. 1). Therefore, $s^{A'} = q$ is an improving deviation for candidate A .

Case 2; $Y \supset \{s^A\}$, i.e. $\exists m \in Y$ s.t. $m \neq s^A$. $[m, s^A] \cap N(s^A) \neq \emptyset$ because any two alternatives are connected via a set of neighbors (in a median space). Let $y \in [m, s^A] \cap N(s^A)$. Because $m \in Y$ and $(m, y, s^A) \in T_{\mathcal{H}}$, it holds that $y \in Y$. Thus, $(x_*^B, y, s^A) \in T_{\mathcal{H}}$, implying that $y \succ^B s^A = \rho(s)$. Moreover, s^A and y differ exactly with respect to one issue—say $y \in H$, $s^A \in H^c$ —(since this holds true for any pair of neighbors). Therefore, $\forall z \in H$ we have $(z, y, s^A) \in T_{\mathcal{H}}$. Thus, single-peakedness implies for a voter i , $y \succ s^A$ if and only if $x_i^* \in H$. This implies that $\rho(s^A, y) = y$ because H^c is a majority property ($q \in H$ means that $w(H) > w(H^c)$ by construction of q). Therefore, $s^{B'} = y$ is an improving deviation for candidate B . ■

For part (ii) of this result, the assumption of heterogeneous preferences is necessary. Consider $x_*^A = x_*^B \neq q$. Then $s^A = s^B = x_*^A$ constitutes a Nash equilibrium. The assumption that a Condorcet winner exists is necessary for the existence of a Nash equilibrium (part i), and for the first case in the proof of part (ii). Reconsider Example 1 for which it holds that $\succeq \in \mathcal{M}$, but

$\succeq \notin \check{\mathcal{M}}$. Let $\succeq^A = (a, b, c, d)$ and $\succeq^B = (b, a, d, c)$. $s^A = b (= \rho(s) = x_*^B)$ and $s^B = d$ constitutes a Nash equilibrium with $\rho(s) \neq a = f(\succeq)$.

The two results show that for the domain $\check{\mathcal{M}}$ both games lead to an implementation of the rule $f(\succeq)$.

5 Discussion

We consider voters whose preferences on a set of alternatives (e.g. political positions) belong to the largest domain of single-peaked preferences for which truthful implementation is possible. By a fundamental result of Nehring and Puppe (2007b), this is the median spaces and the unique rule f of this type (anonymous, neutral, non-dictatorial, and strategy-proof social choice rule) selects the median alternative $f(\succeq)$. Which institutions/mechanisms do implement this rule?

1. By the revelation principle, we can use a direct mechanism: Voters communicate their preferences and f is implemented (by some authority).
2. By the proofs of (Nehring and Puppe 2007b), we can define a binary property space corresponding to the structure of preferences and determine the chosen policy by majority voting on each (political) issue. Letting each agent vote in favor of the property that is possessed by his favorite alternative, f is implemented.
3. If there exists a Condorcet winner (e.g. if the corresponding property space is acyclic), then majority voting on the alternatives themselves also implements f (Prop. 1).
4. If there exists a Condorcet winner, then we can organize a competition between two political candidates in order to implement f (Cor. 1 and Prop. 2).

The fourth possibility can be considered as consisting of the following three stages. Stage 1: Voters announce their preferences (e.g. on political positions) in a poll. Stage 2: Two political candidates compete by choosing an alternative (express a political position). Stage 3: Voters choose a candidate by majority voting. This institution leads to an implementation of the median alternative for the following reasons: In stage 3, there is no incentive to deviate from true preferences since there are only two candidates. In the game played in stage 2, the winning candidate chooses the median alternative $f(\succeq)$ in any Nash equilibrium (Cor. 1 and Prop. 2). In stage 1, voters do not have an incentive to deviate from honesty since rule f is strategy-proof (Nehring and Puppe 2007b).

An interesting aspect of rule f is that it only takes the voters' preference peaks into account. In that sense, the authority (1.) or the competing candidates (4.) need not know the full preferences of the population, but only the distribution of favorite alternatives.

References

- CONGLETON, R. (2002): "The Median Voter Model," in *The Encyclopedia of Public Choice*, ed. by F. Rowley, R. K.; Schneider. Kluwer Academic Publishers.
- DEMANGE, G. (1982): "Single-peaked orders on a tree," *Mathematical Social Sciences*, 3(4), 389–396.
- HANSEN, P., J. F. THISSE, AND R. E. WENDELL (1986): "Equivalence of solutions to network location problems," *Math. Oper. Res.*, 11, 672–678.
- MAS-COLELL, A., M. WHINSTON, AND J. GREEN (1995): *Microeconomic theory*. Oxford University Press New York.
- NEHRING, K., AND C. PUPPE (2007a): "Efficient and strategy-proof voting rules: A characterization," *Games and Economic Behavior*, 59, 132–153.
- (2007b): "The structure of strategy-proof social choice – Part I: General characterization and possibility results on median spaces," *Journal of Economic Theory*, 135(1), 269–305.
- VAN DE VEL, M. (1993): *Theory of Convex Structures*. North-Holland.

Working Paper Series in Economics

recent issues

- No. 17** *Berno Buechel*: A note on Condorcet consistency and the median voter, April 2011
- No. 16** *Eike B. Kroll, Jörg Rieger and Bodo Vogt*: How performance based pay-offs influence estimates of complex information? An experimental study on quality and precision in estimation tasks, March 2011
- No. 15** *Siegfried K. Berninghaus, Thomas Neumann and Bodo Vogt*: Learning in networks - an experimental study using stationary concepts, February 2011
- No. 14** *Marten Hillebrand*: Governmental debt, interest policy, and tax stabilization in a stochastic OLG economy, January 2011
- No. 13** *Arne Beck and Matthias Walter*: Tender prices in local bus transport in Germany - an application of alternative regression techniques, December 2010
- No. 12** *Eike B. Kroll, Holger Müller and Bodo Vogt*: Experimental evidence of context-dependent preferences in risk-free settings, November 2010
- No. 11** *Siegfried Berninghaus, Werner Güth and Stephan Schosser*: Voluntary contributing in a neighborhood public good game - an experimental study, November 2010
- No. 10** *Axel Schaffer, Léopold Simar and Jan Rauland*: Decomposing regional efficiency, October 2010
- No. 9** *Tobias Lindner*: Das Bundestagswahlrecht aus Perspektive der Social Choice-Theorie, October 2010
- No. 8** *Andranik S. Tangian*: Decision making in politics and economics: 4. Bundestag elections 2009 and direct democracy, October 2010
- No. 7** *Martin Ruckes and Thomas Rønde*: Dynamic incentives in organizations: success and inertia, October 2010