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by Jörg Urban

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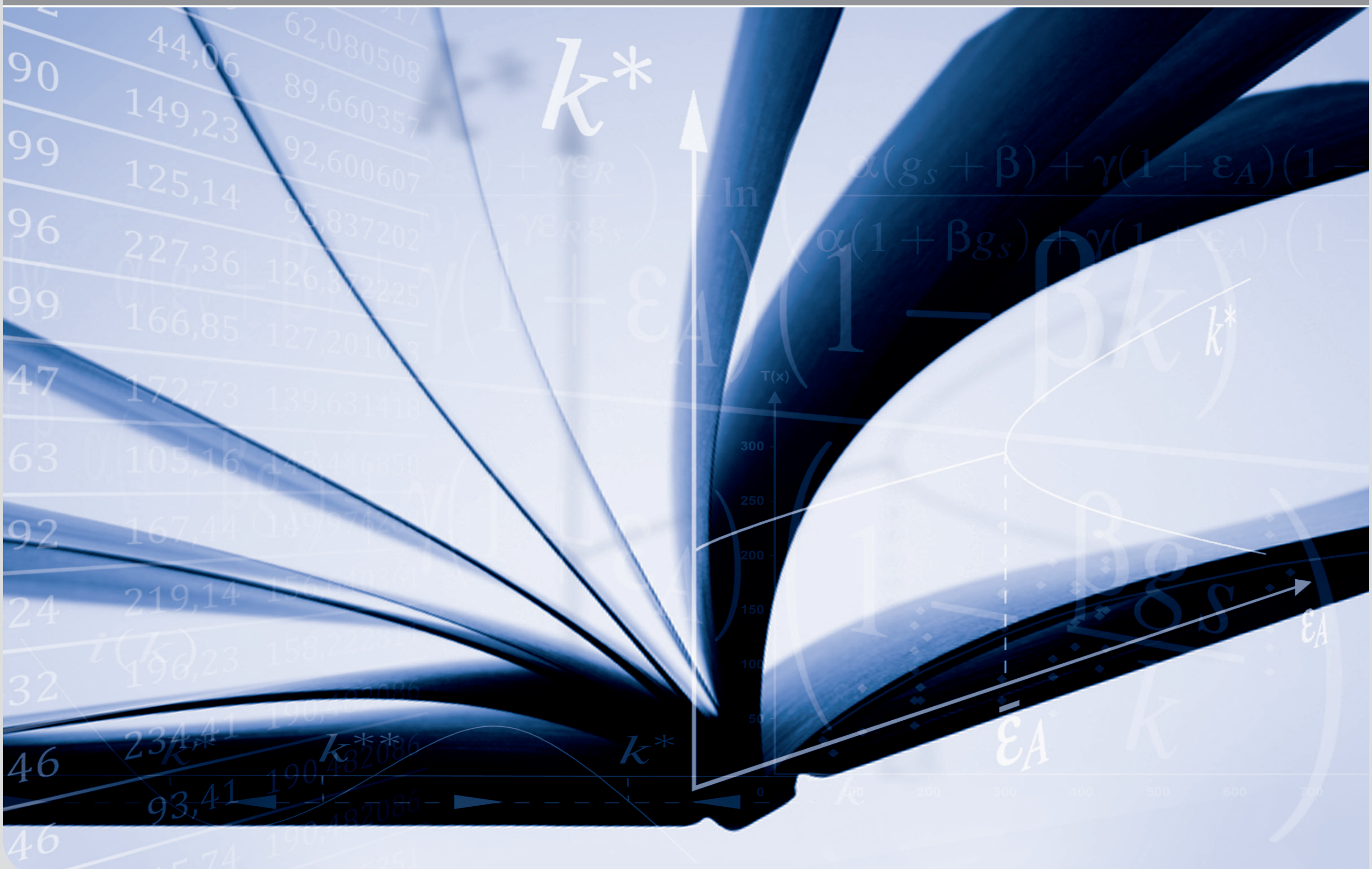

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Credit Cycles revisited*

Jörg Urban[1]

Abstract

Credit and business cycles play an important role in economic research, especially for central banks and supervisors. We reexamine a dynamic model proposed by Kiyotaki and Moore (1997) of an economy with an endogenous credit limit. They claim that a small temporary shock generates large and persistent deviations from the steady state due to a positive feedback loop and the endogenous credit constraint. We mathematically show that contrary to common belief the model does not show amplification and persistence is visible only for a few parameter settings. Kiyotaki and Moore have linearized the model in deviations of landholdings and found that these deviations from the equilibrium are large. This is mathematically inconsistent, because any higher order term would then be more important, rendering any finite-order Taylor expansion invalid. Further, we show that spillover effects in an economy with two distinct sectors are small. The strong amplification present in the original results, which supposedly is due to the large inter-temporal or dynamic multiplier effect, is spurious. The dynamic multiplier effect is of similar size than the static effect and in all cases numerically small.

Keywords: Amplification, credit constraints, credit cycles, dynamic economies, Taylor expansion.
JEL: E32, E37, E51, and E52.

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[1] Karlsruhe Institute of Technology, P.O. Box 6980, 76049 Karlsruhe, Germany; University of Basel, Peter Merian-Weg 6, 4002 Basel, Switzerland. email: joerg.urban@unibas.ch

1. Introduction

Many economists consider credit cycles fundamental drivers to business cycles, eg Fisher (1933) explained the Great Depression with a debt-deflation hypothesis. The latest crisis has moved the topic of credit cycles back into the financial stability focus. A growing debt burden and falling asset prices are not just passive symptoms of an economy in recession, but contribute via feedback loops to a widening of the crisis. Notable work on the persistence of business cycles has been done by Bernanke and Gertler (1989), who formalized these thoughts into a general equilibrium model. In their seminal paper on amplification they show that the condition of borrower's balance sheets is a source of output dynamics. It is believed that the feedback between net worth and investment may lead to a positive amplification during boom and a negative one during bust periods. Several authors have developed dynamic models for closed economies, which try to link credit frictions or constraints to amplification of shocks and persistence, eg Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997), Azariadis and Smith (1998) and Bernanke et al. (1999).

Credit market imperfections have been a major driver for example towards the Great Financial Crisis and also for the East Asian miracle and the subsequent decline. Edison et al. (1998) used the Kiyotaki and Moore (1997) model with highly leveraged firms to analyze the Asian crisis. During the Asian crisis the fall in asset values was followed by liquidity effects causing the land price to fall further, leading to amplification through feedback. Gelos and Werner (1999) examines the Mexican crisis and provide evidence of the financial accelerator mechanism. Rising real estate prices have lowered funding costs and have increased investment activities and higher demand for land. They claim that the reliance of banks on collateral increased the importance of real estate. The financial acceleration is self-enforcing until interrupted by an economy-wide shock. Many models link lending waves to boom-bust cycles and rely on credit channels or the financial accelerator mechanism.

Kiyotaki and Moore (1997) introduced a model of a dynamic economy in which credit limits and asset prices are strongly interlinked: The model and its results are employed and quoted in many recent publications (see for example Guerrieri and Uhlig (2016)). Kiyotaki and Moore (1997) claim that the interactions between credit limits and asset prices turn out to be a powerful transmission mechanism leading to large deviations from the steady state even in the case of small temporary shocks to the economy. The model is built without any means to enforce debt repayment, hence debt must always be secured. The collateralization results in the dual role of a durable asset, which in the model is land, as a factor of production and as collateral. There exists an endogenous credit limit, because land is a constrained resource.

Iacoviello (2005) embedded the mechanism developed by Kiyotaki and Moore (1997) inside a standard New Keynesian general equilibrium model, whereby the collateral capital in this article is real estate. He shows, that collateral constraints increase the response of aggregate demand to house prices.

Another paper related to studies of amplification is Adrian and Boyarchenko (2012). They extend the Kiyotaki-Moore model by introducing state dependent leverage constraints, where the state of the system is a function of current volatility.

The theoretical and empirical work done among others by Kiyotaki and Moore (1997), Carlstrom and Fuerst (1997) and Bernanke et al. (1999) to study the Great Depression, the Asian crisis or the Mexican crisis can be straight forwardly applied to the US sub-prime crisis or the housing boom in Ireland or Spain, which finally caused the euro-area sovereign debt crisis. The run up to the crisis was stamped by increasing real estate prices, investors' herd behavior, banks search for yield and by the unwillingness of regulators, central banks and governments[1] to break that development at an early point. As a result we experienced the largest world-wide recession since the Great Depression, leading to a large number of record bailout programs.

Prior to the US sub-prime crisis real estate prices generally increased. In Ireland, for example, the increase of house prices was 242% from 1992-2005 and in Spain 114% from 1996-2004 (Girouard et al., 2006). The resulting housing bubble allowed many homeowners to refinance their homes at lower interest rates and to finance consumer spending by taking out second mortgages secured by the price appreciation. Central banks lowered their interest rates to encourage borrowing, eg the Federal Reserve lowered the federal funds target rate from 6.5% to 1.0% from January 2001 to June 2003.

The credit expansion was also clearly visible in the balance sheets of financial institutions. European financial institutions accelerated their search for yield and notably expanded their balance sheets in the years before the Lehman default (Nishimura, 2012).

Many academics have extended the Kiyotaki-Moore model, which is consistent with other models incorporating financial acceleration and with the observation of economic cycles. Caballe et al. (2006) uses bifurcation analysis to show that very developed and very undeveloped economies are structurally stable to shocks, while emerging market economies are unstable in the sense that there endogenous variables may exhibit chaotic behavior. Aghion et al. (2000) showed that a currency crisis could emerge when firms are credit constrained and debt is issued in domestic as well as in foreign currency.

The general story of how the Kiyotaki and Moore (1997) model works for a heavily leveraged firm[2] can be summarized as follows: A temporary negative productivity shock at date $t-1$ results in a contemporaneous decline of net worth of the firm, which results in a reduced demand of assets of that firm and ultimately in a decline in asset prices. This within period effect is called the static multiplier. In addition, there exists an inter-temporal multiplier effect: the fall in asset demand of the heavily leveraged firm at date t results in an erosion of funds (ability to borrow) at date $t+1$ which reduces the net worth even more, resulting in a further drop of asset demand hence drop in user costs and asset prices. This inter-temporal effect goes on for any future period $t+1$, $t+2$, ... Kiyotaki and Moore (1997) argue that this dynamic multiplier effect is more powerful than the static effect. The model has some features of a predator-prey model or a positive feedback loop model, with the landholding as the prey and debt as the predator. Kiyotaki and Moore (1997) argue that in a credit constrained economy even a small temporary shock to the production can create large deviations from the steady state due to the dynamic multiplier.

In an extended model with two farming sectors, which are weakly coupled, Kiyotaki and Moore (1997) find that a small shock to only one sector has a large impact on both sectors. They further extend their model ingeniously by introducing heterogeneity amongst farmers and reproducible capital in order to achieve a decoupling of landholding and borrowing. This extended model contains a rich dynamics, eg booms and busts. The authors claim that a small temporary shock generates large and persistent output and

[1] See for example Chancellor G. Brown's speech to the Labour Party Conference in September 1999, where he claimed that booms and busts are abolished.

[2] In the steady state equilibrium of the Kiyotaki-Moore model a firm borrows the maximum amount.

asset price fluctuations. Based on their analysis the following policy statement seems to emerge: Credit restrictions may lead to an amplification of small shocks and to persistence.

These results seem economically intuitive because the strong interlinkage between asset prices and credit limits leads to a positive feedback loop and may lead to persistence and possibly amplification. However, a detailed analysis of the model reveals that the results are spurious, resulting from the linearization of the equations of motion (EOM). Our analysis can be summarized as follows: The model introduced by Kiyotaki and Moore (1997) behaves mathematically well, in the sense that small shocks have only a small impact on the landholding and the static multiplier effect is of similar importance as the inter-temporal/dynamic multiplier in the basic model. Our results follow from an exact solution of the EOM. Further, an economy with two weakly connected farming sectors, where a small shock is only applied to one sector shows only a weak response to the shocked sector and, as expected, a weaker response to the second sector. The full model shows persistence for some parameter settings, but also no amplification. As a conclusion, we can say that credit constraints do not necessarily act as a shock enhancer.

Amplification or over-exaggeration in economic systems should only appear if the state of the economy is in a bubble and a small disturbance is breaking this bubble. Asset prices and government debt have been in such a state near the end of the first decade of the 21st century. The crisis has not been amplified simply by a positive feedback, but because the state of the economy was unsustainable. Further means of amplification can be found in behavioral economics. The simple fear of a crisis can lead to non-rational behavior and for example bank runs, which may act as a strong amplification effect. Gertler and Kiyotaki (2015) have found that small negative shocks by themselves do not produce an amplified effects on production, but open up the possibility of bank-runs, which may lead to devastating outcomes.

In a real economy, asset prices may not be priced "correctly" due to speculation, market friction, limited information or liquidity. The Kiyotaki-Moore model does not include the possibility of a miss pricing of assets or the building of bubbles. Despite the model's theoretical beauty it cannot describe any real post-crisis dynamics. However, as we will see later, the model can show certain stylized facts with respect to a credit crisis.

Reading post-financial crisis literature one gets the impression, that supervisors and central banks are starting to understand the sources of what the path towards a crisis looks like (eg on the amplification of credit cycles see FSF-BCBS Working Group (2009)). In fact, the same knowledge has already been present in the public domain prior to the crisis. Crockett (2000) has already discussed that the path towards a crisis contains some shared stylized elements: Asset prices are surging linked with rapid credit expansion and leverage accumulation in the balance sheets. Similar arguments are brought forward by Borio (2006). This brings us straight back to the impressive analysis of Krugman (2009) and finally to Kiyotaki and Moore (1997) and makes us wonder: Why was nobody worried seeing house prices rise by 10% year after year or balance sheets expand unsustainably?

As a key contribution, we correct the solution presented in Kiyotaki and Moore (1997) and show that the Kiyotaki-Moore model exhibits, unlike so far believed, no amplification and persistence is limited to certain parameters in the full model only. A linearization in powers of deviations of the landholding from the steady state can only be applied for small deviations. The linearized solution in Kiyotaki and Moore (1997) suggests large deviations of the landholding, hence any higher order term of the landholding is more (not just) important and hence not negligible. Further, the linearized solution (Kiyotaki and Moore, 1997) shows a singular behavior in a zero interest rate environment (ie for $R = 1$). Solving the model correctly, we show that the huge amplifications are spurious. Collateral constraints do not act as a powerful amplification and propagation mechanism of a temporary exogenous shock in this model, ie the dynamic multiplier effect turns out to be small. This finding is also consistent with the results found by studying other models (see eg Cordoba and Ripoll (2004)). Krugman (2009) wrote a well-thought analysis about the state of economics and among other things argued about the danger of impressive-looking mathematics. Mathematical models do never mirror reality perfectly. Nevertheless, mathematics is the most powerful tool researchers of all kinds have at their disposal. However, we have to solve models correctly, apply mathematical tools appropriately and have to understand when a model breaks down, ie when the situation we want to describe is outside the scope of the model.[3]

The paper is organized as follows: In Section 2 we analyze Kiyotaki-Moore's basic model. Sections 3 and 4 extend the analysis to the two-sector model and the full model (containing cycles and investment). Section 5 concludes. In order to keep the article self-contained, we have briefly reproduced the steady state solutions and the linear results of Kiyotaki and Moore (1997) in the Appendix.

2. The Basic Model

2.1. The Characteristics of the Basic Model

The basic model introduced by Kiyotaki and Moore (1997) contains a durable asset (land), which can be used as collateral, and a non-durable commodity (fruits), which is used as numeraire and grows on land. There are two types of infinitely lived agents: farmers and gatherers, which are both risk neutral. The population sizes of the farmers and gatherers are normalized to 1 and m, respectively. There exists a competitive spot market where land can be exchanged for fruits and a one-period credit market in which one unit of fruit is exchanged for a claim to R_t units of fruit in the next period.

The farmer has a constant return to scale production function: $y_{t+1} = F(k_t) = (a+c)k_t$, where k_t is the land use and y_t is the output. ak_t represents the tradable output and ck_t the non-tradable output (bruised fruits, which can be used for consumption only). The rationale behind the introduction of c is to avoid that farmers constantly postpone consumption in favor for investment.

The model does not include any aggregate uncertainty, ie agents have perfect foresight over the future land price. b_t, the amount a farmer can borrow, is restricted by the following condition $Rb_t \leq q_{t+1}k_t$, where q_{t+1} is the land price in the future period.[4] Furthermore, in this model the farmer can only expand the scale of production by investing in more land.

[3] One can use Newton's equations of gravity to describe the behavior of an apple in the gravitational field of the earth, but the movement of a star near a black hole requires Einstein's General Theory of Relativity.

[4] $q_{t+1}k_t$ is the liquidation value (outside value) of the land for the creditor (gatherer). In equilibrium, farmers borrow from gatherers. The interest rate equals the inverse of the gatherers' constant discount factor, ie $R_t = 1/\tilde{\beta} \equiv R$. The farmer can borrow up to the next periods' value of its current landholding.

The farmers' flow of fund constraint is

$$\overbrace{q_t(k_t - k_{t-1}) + Rb_{t-1} + \underbrace{x_t - ck_{t-1}}_{\text{consumption above automatic}}}^{\text{cash out}} = \overbrace{ak_{t-1} + b_t}^{\text{cash in}}\,, \tag{1}$$

with farmer's consumption x_t, the accumulated debt including interest Rb_{t-1} and the investment in more land $q_t(k_t - k_{t-1})$.

For all gatherers we assume an identical decreasing returns to scale production function $\tilde{y}_{t+1} = G(\tilde{k}_t)$ with $G' > 0$ and $G'' < 0$. Variables with a tilde refer to gatherers.

The gatherers' flow of fund constraint is

$$\overbrace{q_t(\tilde{k}_t - \tilde{k}_{t-1}) + R\tilde{b}_{t-1} + \tilde{x}_t}^{\text{cash out}} = \overbrace{G(\tilde{k}_{t-1}) + \tilde{b}_t}^{\text{cash in}}\,, \tag{2}$$

where $\tilde{b}_t < 0$, because gatherers are creditors.

The model contains several technical assumptions: *A1:* the farmer is relatively impatient, ie the farmers' discount factor is smaller than the gatherers' discount factor: $\beta < \tilde{\beta}$. The rationale behind this assumption is that in equilibrium it is better for the farmer to invest rather than repay debt. *A2:* $\beta > a/(a + c)$, which motivates the farmer not to consume more than c. *A3:* Exploding bubbles are ruled out in the model through the assumption $\lim_{s\to\infty} E_t(R^{-s}q_{t+s}) = 0$. *A4:* In order to assure that gatherers operate close to the steady state equilibrium, we assume: $G'(0) > aR > G'(\overline{K}/m)$, where $\overline{K}$ denotes the total constant land supply.

The equilibrium is characterized by the tuple $\{q_t, k_t, \tilde{k}_t, b_t, \tilde{b}_t, x_t, \tilde{x}_t\}$, where farmers and gatherers maximize their expected utility $E_t(\sum \beta^s x_{t+s})$ and $E_t(\sum \tilde{\beta}'^s \tilde{x}_{t+s})$, respectively. Kiyotaki and Moore (1997) show that for farmers it is strictly better to invest than to save and to save is better than to consume. Therefore, in the equilibrium of the basic model farmers borrow the maximum possible amount and consume only the bruised fruits, ie

$$Rb_t = q_{t+1}k_t \qquad \text{and} \qquad x_t = ck_{t-1}\,. \tag{3}$$

For more detail on the basics of the model, we refer the reader to the well written paper Kiyotaki and Moore (1997).

A trivial note on Taylor-expansion and linearization

The Taylor-series is an approximation method with the aim to produce a locally good approximation of a function (see for example Judd (1998)). Assuming that we deal with C^∞ functions, ie functions which are infinitely differentiable, then we can write the function as a Taylor series around a:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \approx f(a) + \frac{f'(a)}{1!}(x-a),$$

where the last part represents the linearization. We will later rewrite the models' equations of motion in terms of deviations from the steady state and hence $a = 0$ in our case. Therefore, we are analyzing Maclaurin series. A function is called analytic in an open set D, if the coefficients of the series are $\in \mathbb{R}$ and the series is convergent to $f(x)$ for all x in a neighborhood of $a \in D$. The Taylor expansion is often used to linearize or approximate problems which are otherwise unsolvable. Very often, in economics small shocks Δ are applied and then the function f is linearized:

$$f(\Delta) = f(0) + \frac{f'(0)}{1!}\Delta + \mathcal{R}(\Delta) = f(0) + \frac{f'(0)}{1!}\Delta + \frac{f''(x^*)}{2!}\Delta^2,$$

where $\mathcal{R}(\Delta)$ is the remainder of the first order Taylor expansion. For practical purpose, we have assumed that the function is well behaved, so that the remainder can be written in its mean-value form with some $x^* \in [0,\, \Delta]$. For sufficiently small Δ clearly the linearization is a good approximation. If we apply a small shock and do not expand in Δ but in powers of other model parameters, such as deviations of the landholding, debt holding or price, we have to ensure that these deviations are small enough that the remainder term is negligible. Otherwise, the linearization becomes poor or even invalid. We will show next that this is what happens for the classical solution of the KM model.

2.2. The Equations of Motion of the Basic Model

The structure of Equations (1) and (3) with respect to k_t and b_t allows to compute aggregate EOM:

$$K_t = \frac{1}{u_t}\left[(a+q_t)K_{t-1} - RB_{t-1}\right] \qquad \text{and} \qquad B_t = \frac{1}{R} q_{t+1} K_t\,, \tag{4}$$

where $u_t \equiv q_t - q_{t+1}/R$ can be interpreted as the required down payment for one unit of land.

Gatherers are not credit-constrained and their demand for land is determined through the condition that the present value of the marginal product is equal to the opportunity cost for one unit of land (maximizing the gatherers' expected utility using the gatherers' fund of flow constrained in Equation (2)), ie

$$\frac{1}{R}G'(\tilde{k}_t) = q_t - \frac{1}{R}q_{t+1} = u_t\,. \tag{5}$$

It is obvious from the last equation that the term u_t plays a dual role in the model as down payment and opportunity cost.

The technology $a+c$ is considered time invariant. We will vary a only in form of a small temporary shock to study the model's dynamics. The price of land q_t is always linked directly to the user cost. In a real economy, speculators may bet on future productivity and higher output and therefore drive the land price away from fundamentals. This may result in bubbles, which are not included in this model.

We agree with the steady state solution presented in Kiyotaki and Moore (1997) and have reproduced it in Appendix A. Here we focus on the dynamics of the model. We assume that the model is in equilibrium at $t-1$ and a temporary shock Δ is applied to the productivity at t. From Equations (4) the non-linear EOM follow:

$$\begin{aligned} u(K_t)K_t &= (\overbrace{a+\Delta a}^{\text{due to shock}} + \overbrace{q_t - q^*}^{\text{capital gain due to price jump}})K^* && \text{at } t\,, && (6)\\ u(K_{t+s})K_{t+s} &= aK_{t+s-1} && \text{at } t+s \text{ with } s \geq 1\,. && (7)\end{aligned}$$

The net worth at t increases due to the direct productivity effect from the shock and the capital gain effect of the price jump. The equation for $t+1$ onwards states that after the shock the farmers can hold land up to a point where the required down payment is covered by its net worth which is a (see aggregate EOM (4)).

2.3. The response to a small temporary shock

In order to analyze the model, we define relative deviations from the steady state as $\hat{X}_t = (X_t - X^*)/X^*$, where X_t is B_t, K_t or q_t. Variables with a star denote the value at the steady state. Kiyotaki and Moore (1997) suggest to solve the Equations (6) and (7) for a small temporary shock Δ by linearizing around the steady state (see Appendix A). We start with Equation (6) and insert the Taylor expansion of $u(K_t)$ around the steady state:

$$u(K_t) = u^* + \hat{K}_t K^* u'(K^*) + \frac{1}{2}\hat{K}_t^2 K^{*2} u''(K^*) + \mathcal{O}(\hat{K}_t^3) = u^* \left(1 + \frac{1}{\eta}\hat{K}_t + \frac{1}{\rho}\hat{K}_t^2 + \mathcal{O}(\hat{K}_t^3)\right), \tag{8}$$

where $u^* \equiv u(K^*)$, $1/\eta \equiv K^*/u^*\, u'(K^*)$[5] and $1/\rho \equiv K^{*2}/(2u^*)\, u''(K^*)$, resulting in

$$\left(1 + \frac{1}{\eta}\right)\hat{K}_t + \left(\frac{1}{\eta} + \frac{1}{\rho}\right)\hat{K}_t^2 + \mathcal{O}(\hat{K}_t^3) = \Delta + \frac{R}{R-1}\hat{q}_t\,. \tag{9}$$

In order to arrive at Equation (9), we have applied the steady state Equations (A.2) and (A.3).

Similarly, we find for Equation (7):

$$\left(1 + \frac{1}{\eta}\right)\hat{K}_{t+s} + \left(\frac{1}{\eta} + \frac{1}{\rho}\right)\hat{K}_{t+s}^2 + \mathcal{O}(\hat{K}_t^3) = \hat{K}_{t+s-1}\,. \tag{10}$$

From Equation (A.1) and the assumption that we have no exploding price bubbles, we know $q_t = \sum R^{-s} u(K_{t+s})$. Therefore, using Equation (8) we can write the equation for the land price:

$$\hat{q}_t = \frac{R-1}{R}\left(\frac{1}{\eta}\sum_{s=0}^{\infty} R^{-s}\hat{K}_{t+s} + \frac{1}{\rho}\sum_{s=0}^{\infty} R^{-s}\hat{K}_{t+s}^2 + \mathcal{O}(\sum_{s=0}^{\infty} R^{-s}\hat{K}_{t+s}^3)\right)\,. \tag{11}$$

In order to make the computation feasible, we assume as in Kiyotaki and Moore (1997) the functional form $u(K) = K - \nu$.[6] This expression is the simplest possible extension beyond a trivial constant, and still we can already show that a linearization yields incorrect results when large output deviations are studied.

Firstly, we observe that in Equations (9) to (11) all terms involving ρ disappear because $u'' = 0$ and secondly the $\mathcal{O}(\hat{K}_t^3)$ terms are gone as well. The system of equations

[5] This definition is identical to definition in footnote 14 in Kiyotaki and Moore (1997).

[6] It is evident that $u(K) > 0$ close to the steady state K^* and via the market clearing condition in Equation (A.1) we can conclude that $G'(\tilde{k}) > 0$ as assumed for the gatherer's production function. Further, from $u'(K) = du/dK = 1 > 0$ and via the market clearing condition in Equation (A.1):

$$u'(K) = \frac{d}{dK}\frac{1}{R}G'(\tilde{k}) = \frac{1}{R}\frac{d\tilde{k}}{dK}\frac{d}{d\tilde{k}}G'(\tilde{k}) = \frac{1}{R}\frac{d(\overline{K}-K)}{m\,dK}\frac{d}{d\tilde{k}}G'(\tilde{k}) = -\frac{G''(\tilde{k})}{Rm} = 1$$

we have $G''(\tilde{k}) < 0$, again as assumed for the gatherer's production function.

is, as the whole model, non-linear in $\hat{K}_t$. If we neglect non-linear terms, respective linearizing the EOM, we reach the simple Equations (A.6) to (A.8). Solving the non-linearized system exactly is cumbersome and yields unmanageable expressions. We prefer a numeric solution, which is simply achieved using eg MatLab or any other program. For the numerical illustration, we set as in Kiyotaki and Moore (1997) η equal to 10%, ie $\eta = 0.10$. We vary the shock Δ between 1% and 5% and R between 1.00 and 1.05. The response of the landholding and land price based on the linearized solution, as derived in Kiyotaki and Moore (1997) and reproduced in Appendix A, are presented in Table 1. The responses based on an exact solution are shown in Table 2.

Table 1: Response of the landholding and land price to a temporary shock Δ, linearized approximation

This table reports the relative deviations of the landholding from the steady state at time t for a variety of different shock sizes and interest rates as well as the relative deviations of the land price (which are independent of R) based on the linearization of the EOM. η was chosen 0.1. We find large deviations of K_t from the steady state, ie in most cases $\hat{K}_t$ is of order 1 and hence $\hat{K}_t >> \Delta$.

Δ	0.01	0.03	0.05
$\hat{K}_t$ at $R = 1.00$	∞	∞	∞
$\hat{K}_t$ at $R = 1.01$	0.9191	2.7573	4.5955
$\hat{K}_t$ at $R = 1.03$	0.3130	0.9391	1.5652
$\hat{K}_t$ at $R = 1.05$	0.1918	0.5755	0.9591

Δ	0.01	0.03	0.05
$\hat{q}_t$	0.1000	0.3000	0.5000

The numbers for the response of the landholding to a small temporary shock based on the linear approximation in Table 1 clearly illustrate that even a small shock leads to large changes in the landholding. For $R = 1$ (zero interest rate) the linear solution exhibits a singularity, which is due to the factor $1/(R-1)$ (see Equation (A.9)). These results clearly indicate that a finite order Taylor expansion in the landholding is incorrect, because any higher order term in $\hat{K}_t$ is more (not just) important. Clearly, the solution in Table 2 is completely different to the linearised solution in Table 1, especially the singularity at $R = 1$ has disappeared. One could be tempted to ignore the huge differences between Tables 1 and 2 and by "cherry-picking" argue that there is still amplification. However, Kiyotaki and Moore noticed themselves that the basic model is too simplistic to be of practical relevance. In order to achieve pertinence, we have to look at Kiyotaki and Moore's more realistic full model (see Section 4), in which they have added reproducible capital and investment opportunities to study cycles.

Table 2: Response of the landholding and land price to a temporary shock Δ, exact solution

This table reports the relative deviations of the landholding and land price from the steady state at time t for a variety of different shock sizes and interest rates based on the exact solution using $u(K) = K - 9u^*$ ($\eta = 0.1$). Unlike in Table 1, small temporary shocks result in small deviations from the steady state. The deviation of the price from the equilibrium depends now on Δ and R, unlike in the linearized case.

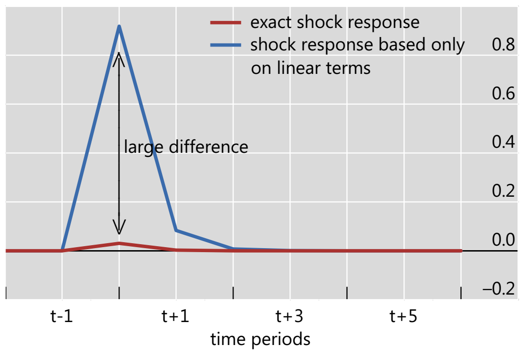
Δ	0.01	0.03	0.05
$\hat{K}_t$ at $R = 1.00$	0.0315	0.0545	0.0704
$\hat{K}_t$ at $R = 1.01$	0.0310	0.0540	0.0699
$\hat{K}_t$ at $R = 1.03$	0.0300	0.0530	0.0689
$\hat{K}_t$ at $R = 1.05$	0.0290	0.0520	0.0679

Δ	0.01	0.03	0.05
$\hat{q}_t$ at $R = 1.00$	0.0000	0.0000	0.0000
$\hat{q}_t$ at $R = 1.01$	0.0034	0.0059	0.0076
$\hat{q}_t$ at $R = 1.03$	0.0096	0.0169	0.0220
$\hat{q}_t$ at $R = 1.05$	0.0151	0.0271	0.0354

The evolution over time of the response of the landholding to a small temporary shock $\Delta = 0.01$ for $R = 1.01$ and $\eta = 0.1$ is presented in Figure 1 for the linearized and the exact result.

Figure 1: Evolution of shock responses of the landholding, linear approximation versus exact computation

The figure illustrates the huge difference between the exact result derived here and the linearized result given in Kiyotaki and Moore (1997). The figure is generated using $\Delta = 0.01$, $R = 1.01$ and $\eta = 0.1$.



From Figure 1 we can clearly see that the basic model does not show persistence, ie the system returns quickly back to the steady state. Further, consistent with Table 2, a small temporary shock does not produce large deviations from the steady state (see red curve in Figure 1) in the presence of credit constraints if the model is solved exactly. We find the same behavior also for the land price.

2.4. Static versus dynamic multiplier effect

In this section we aim to understand how much of the instantaneous shock response is due to the static and how much is due to the dynamic multiplier effect. Similar to Kiyotaki and Moore (1997), we perform the following thought experiment: We peg q_{t+s}

artificially at q^* in order to simulate the static multiplier effect only.[7] The difference of the full solution and the static effect will then give us the dynamic multiplier effect. Equation (11) gets simplified because only the first term in the two sums is considered:

$$\hat{q}_t\Big|_{q_{t+1}=q^*} = \frac{R-1}{R}\left(\frac{1}{\eta}\,\hat{K}_t\Big|_{q_{t+1}=q^*} + \frac{1}{\rho}\,\hat{K}_t^2\Big|_{q_{t+1}=q^*} + \mathcal{O}(\hat{K}_t^3)\right). \tag{12}$$

In order to keep expressions analytically manageable we hold on to the special form of $u(K) = K - 9u^*$. Inserting Equation (12) into Equation (9) we find:

$$\left(1+\frac{1}{\eta}\right)\hat{K}_t\Big|_{q_{t+1}=q^*} + \frac{1}{\eta}\,\hat{K}_t^2\Big|_{q_{t+1}=q^*} = \Delta + \frac{1}{\eta}\,\hat{K}_t\Big|_{q_{t+1}=q^*}. \tag{13}$$

Ignoring the quadratic term in Equation (13) as done in Kiyotaki and Moore (1997) leads for the static multiplier effect to:

$$\hat{K}_t\Big|_{q_{t+1}=q^*}^{\text{linear}} = \Delta \qquad \text{and} \qquad \hat{q}_t\Big|_{q_{t+1}=q^*}^{\text{linear}} = \frac{R-1}{R}\frac{1}{\eta}\Delta\,. \tag{14}$$

Including the quadratic term, we find two solutions for the landholding. The economic solution, which must be zero in the limit of $\Delta \to 0$, is:

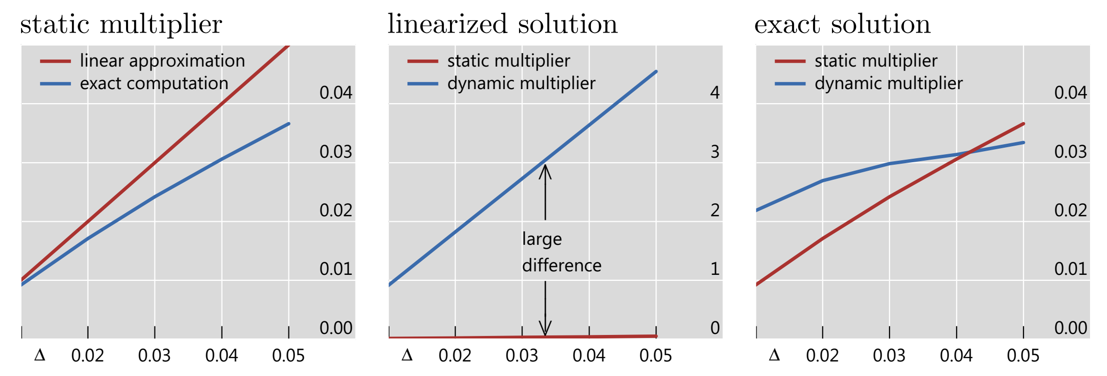
$$\hat{K}_t\Big|_{q_{t+1}=q^*}^{\text{exact}} = -\frac{\eta}{2} + \sqrt{\frac{\eta^2}{4} + \eta\Delta} \qquad \text{and} \qquad \hat{q}_t\Big|_{q_{t+1}=q^*}^{\text{exact}} = -\frac{R-1}{R}\left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\Delta}{\eta}}\right). \tag{15}$$

The difference of the static multiplier effect for the exact computation in Equation (15) and the linear approximation in Equation (14) is rather small (see left hand panel of Figure 2). Kiyotaki and Moore (1997) have claimed that the dynamic or inter-temporal multiplier effect is much stronger than the static effect. Indeed, the middle panel of Figure 2, which was derived from the linearized solution (figures in Table 1 and left-hand Equation (14)), suggests that the dynamic multiplier effect is much larger than the static response. From our recent analysis, we know that the linearization is invalid, even for small shocks. The difference of the static multiplier effect and the dynamic multiplier effect is rather small if the model is solved correctly (figures in Table 2 and left-hand Equation (15)). The dynamic multiplier effect is of similar size as the static effect and not dominant at all, as can be seen in the right-hand panel in Figure 2. We find a similar picture for the land price.

[7] As defined in Kiyotaki and Moore (1997), any impact to and from the future is ignored in the static multiplier effect (within period effect).

Figure 2: Static versus dynamic multiplier effect as function of Δ for the landholding

The figure illustrates shock responses for $R = 1.01$ and $\eta = 0.1$. The horizontal axis shows the size of the shock, starting at a 1% shock. The left-hand panel compares the static multiplier effect derived via linearization and exact computation. It shows that the results using the linear approximation and the exact computation are similar. The middle panel shows the results presented in Kiyotaki and Moore (1997), where the dynamic multiplier is much larger than the static multiplier. The right-hand panel shows the exact computation in which both multiplier effects are of similar size.



3. The Basic Model extended with two coupled sectors

3.1. The Characteristics of the Extended Model

The basic model studied so far does not allow to analyze spillovers between sectors. Therefore, we follow Kiyotaki and Moore (1997) and add a second farming sector, where farmers in sector i =1 or 2 produce $a_i k_{it-1}$ sector specific fruits and $c_i k_{it-1}$ regular fruits (for consumption). Gatherers produce regular fruits and have the same production function as in the basic model. The sectors are indirectly linked via the land price q_t. The direct interlinkage of the sectors is given through the following assumed equivalence of consumption bundles: $x_t^{1-\epsilon} = x_{1t}^{1-\epsilon} + x_{2t}^{1-\epsilon}$, where $\epsilon > 0$ is the inverse of the elasticity of substitution (constant elasticity of substitution) in consumption between the two types of fruits. x_{it} is the consumption of fruits from sector i and x_t is the consumption of regular fruits. Regular fruits are considered as a numeraire. The competitive price p_{it} for sector i fruits in terms of regular fruits is equivalent to the marginal rate of substitution:

$$p_{it} = (a_i K_{it-1})^{-\epsilon} \left[(a_1 K_{1t-1})^{1-\epsilon} + (a_2 K_{2t-1})^{1-\epsilon}\right]^{\epsilon/(1-\epsilon)} , \qquad (16)$$

where K_{it-1} is the aggregate landholding of farmers in sector i. Further details on the characteristics of the extended model can be found in Kiyotaki and Moore (1997).

3.2. The Equations of Motion of the Extended Model

The aggregated EOM for sector i look very similar to the basic model. However, the price of tradable fruits for sector i cannot be normalized to one anymore, ie the production a_i needs to be multiplied by the competitive price p_{it}:

$$K_{it} = \frac{1}{u_t}\left[(a_i p_{it} + q_t)K_{it-1} - RB_{it-1}\right] \qquad \text{and} \qquad B_{it} = \frac{1}{R} q_{t+1} K_{it} \, . \qquad (17)$$

The land market equilibrium is given by:

$$u_t = q_t - \frac{1}{R} q_{t+1} = u(K_{1t} + K_{2t}) \,, \tag{18}$$

stating that the land price depends on the entire landholding $K_{1t} + K_{2t}$.

Again, we focus on the dynamics of the model and apply a small temporary shock Δ to the productivity of sector $i = 1$, only. We assume that the entire system is in equilibrium at $t-1$ (the solution of the system in equilibrium is reproduced in Appendix B). Further, as in Kiyotaki and Moore (1997) we analyze a symmetric model, ie we have equal productivity as well as land- and debt-holdings in equilibrium: $a_1 = a_2 = a$, $K_1^* = K_2^* = K^*/2$ and $B_1^* = B_2^* = B^*/2$. The EOM at t can be written as (using $u^* = a2^{\epsilon/(1-\epsilon)}$, see Appendix B):

$$\begin{aligned} u_t K_{1t} &= \left[u^*(1 - \frac{\epsilon}{2}\Delta)(1+\Delta) + q_t - q^* \right] K_1^* \\ &= \left[u^* + (1 - \frac{\epsilon}{2}) u^* \Delta + q_t - q^* \right] K_1^* \,, \qquad (19) \\ u_t K_{2t} &= \left[u^* + \frac{\epsilon}{2} u^* \Delta + q_t - q^* \right] K_2^* \,, \qquad (20) \end{aligned}$$

where we have taken only the linear term in Δ.[8] In addition, we have substituted the Taylor expansion up to the first order in Δ of the competitive prices of sector specific fruits p_{it} (Equation (16)):

$$\begin{aligned} p_{it} &= ((a + a\Delta\delta_{i1})K_i^*)^{-\epsilon} \left[((a + a\Delta)K_1^*)^{1-\epsilon} + (aK_2^*)^{1-\epsilon} \right]^{\epsilon/(1-\epsilon)} \\ &= (1 + \Delta\delta_{i1})^{-\epsilon} ((1+\Delta)^{1-\epsilon} + 1)^{\epsilon/(1-\epsilon)} \\ &= 2^{\epsilon/(1-\epsilon)} \left(1 + (-1)^i \frac{\epsilon}{2} \Delta \right) \,, \qquad (21) \end{aligned}$$

where the Kronecker-delta is $\delta_{11} = 1$ and $\delta_{21} = 0$.

The EOM for both sectors at $t+s$ with $s \geq 1$ look like:

$$u_{t+s} K_{it+s} = a p_{it+s} K_{it+s-1} = u^* \frac{p_{it+s}}{2^{\epsilon/(1-\epsilon)}} K_{it+s-1} \,. \tag{22}$$

3.3. The response to a small temporary shock to sector $i = 1$

In order to analyze the model, we rewrite the system of non-linear Equations (19), (20) and (22) in terms of relative deviations from the steady state, ie $\hat{K}_{it} = (K_{it} - K_i^*)/K_i^*$ for $i = 1, 2$:

$$\frac{u_t(\hat{K}_{it} + 1)}{u^*} - 1 = \left(\delta_{i1} + (-1)^i \frac{\epsilon}{2} \right) \Delta + \frac{R}{R-1} \hat{q}_t \,. \tag{23}$$

and for $t+s$ with $s \geq 1$:

$$\frac{u_{t+s}(\hat{K}_{it+s} + 1)}{u^*} = \frac{p_{it+s}}{2^{\epsilon/(1-\epsilon)}} (\hat{K}_{it+s-1} + 1) \,. \tag{24}$$

Kiyotaki and Moore (1997) have linearized the system (see Appendix B). Our aim is to compare their results with the exact solution. We start first with instantaneous shock

[8] In this case the linearization is valid, because we know that Δ is small by assumption.

response where as in Section 2.3 we assume $u(K_1 + K_2) = K_1 + K_2 - \nu$. This functional form is the simplest, beyond a trivial constant, but already allows to make our point clear that a Taylor expansion in landholdings is unsuitable if large deviations of landholdings from the steady state are studied. Under the assumption of the special form of $u(K_1+K_2)$ the power series in $\hat{K}_{it}$ around the steady state $K^* = K_1^* + K_2^*$ is exact:

$$\hat{K}_{it} + \frac{\hat{K}_{1t} + \hat{K}_{2t}}{\eta_i}\left(1 + \hat{K}_{it}\right) = \left(\delta_{i1} + (-1)^i \frac{\epsilon}{2}\right)\Delta + \frac{R}{R-1}\hat{q}_t, \tag{25}$$

$$\hat{K}_{it+s} + \frac{\hat{K}_{1t+s} + \hat{K}_{2t+s}}{\eta_i}\left(1 + \hat{K}_{it+s}\right) = \frac{p_{it+s}}{2^{\epsilon/(1-\epsilon)}}\left(\hat{K}_{it+s-1} + 1\right) - 1\,. \tag{26}$$

where $1/\eta_i = 1/(2\eta) = K^*/(2u^*)u'(K^*)$.

Solving the linearised solution is straight forward (see Kiyotaki and Moore (1997) and Appendix B), but for the exact solution we need to stick to a numerical analysis. As previously we take $\eta = 0.1$, vary R between 1.00 and 1.05 and the temporary shock Δ between 0.01 and 0.05. Like Kiyotaki and Moore (1997) we take $\epsilon > 0$ but small (weakly coupled), eg $\epsilon = 0.5$. The deviation of the landholding from the equilibrium for different R and shocks Δ are presented in Table 3 for the linear approximation and in Table 4 for the exact solution.

Table 3: Response of the landholding to a temporary shock Δ, linear solution

This table reports the relative deviations of the landholding for both sectors from the steady state at time t for a variety of different shock sizes and interest rates, based on the linearization. As an exemplification, η and ϵ were chosen 0.1 and 0.5, respectively.

	$\hat{K}_{1t}$			$\hat{K}_{2t}$		
Δ	0.01	0.03	0.05	0.01	0.03	0.05
$R = 1.00$	∞	∞	∞	∞	∞	∞
$R = 1.01$	0.4620	1.3861	2.3102	0.4570	1.3711	2.2852
$R = 1.03$	0.1590	0.4770	0.7951	0.1540	0.4620	0.7701
$R = 1.05$	0.0984	0.2952	0.4920	0.0934	0.2802	0.4670

In Table 3, we see that a small temporary shock results in large instantaneous deviation from the equilibrium, which makes the proposed linearization invalid. Further, both sectors are similarly affected, despite the fact, that sector 1 is shocked only. Moreover, the solution presented in Kiyotaki and Moore (1997) exhibits again a pole at $R = 1$, ie for zero interest rates the deviations from the equilibrium is infinitely large independent of the shock size. Table 4 shows that these large deviations are spurious.

Table 4: Response of the landholding to a temporary shock Δ, exact solution

This table reports the relative deviations of the landholding for both sectors from the steady state at time t for a variety of different shock sizes and interest rates, based on the non-linear approach. η and ϵ were chosen 0.1 and 0.5, respectively.

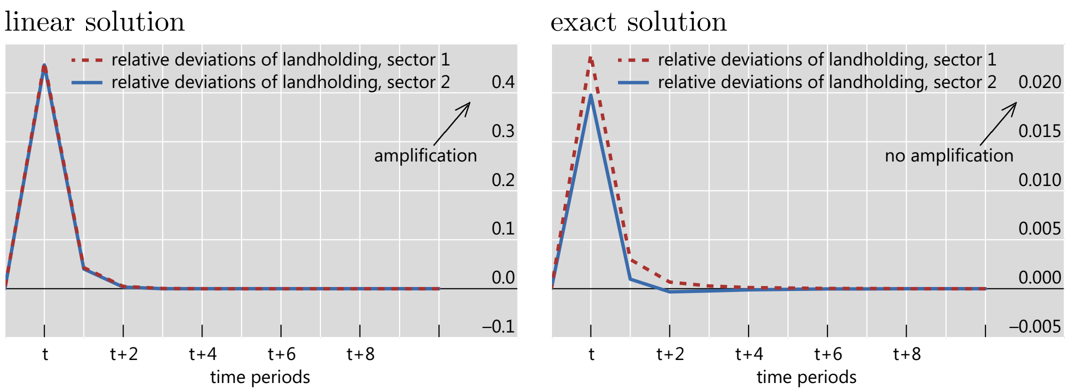
	$\hat{K}_{1t}$			$\hat{K}_{2t}$		
Δ	0.01	0.03	0.05	0.01	0.03	0.05
$R = 1.00$	0.0244	0.0441	0.0583	0.0203	0.0333	0.0416
$R = 1.01$	0.0239	0.0436	0.0578	0.0198	0.0328	0.0411
$R = 1.03$	0.0229	0.0426	0.0568	0.0187	0.0317	0.0400
$R = 1.05$	0.0220	0.0417	0.0559	0.0178	0.0307	0.0390

The exact solution in Table 4 shows the desired behavior, namely that the deviations are of the same order of magnitude as the shock. Table 4 also shows that the shock to the first sector has a smaller impact on the second sector.

The evolution over time of the system after a small temporary shock is plotted in Figure 3.

Figure 3: Evolution of shock responses of the landholding, linear approximation versus exact computation

The figure illustrates the huge difference between the exact results and the linearized results given in Kiyotaki and Moore (1997). The figure is generated using $\Delta = 0.01$, $R = 1.01$, $\eta = 0.1$ and $\epsilon = 0.5$.



The linearized solution shows in Figure 3 large deviations from zero and practically no differences in shock responses of the two sectors. On the other hand, the deviation from the steady state, based on the exact solution (right-hand panel in Figure 3), is in the order of the shock size and there is a visible difference in the responses of the two sectors. The second sector (unshocked sector) turns slightly negative. The negative deviations (also for the linearized solution) become stronger with decreasing ϵ.[9] We find no persistence.

[9] ϵ is the inverse of the elasticity of substitution and is greater 0.

3.4. Static versus dynamic multiplier effect

As part of the analysis of the extended model, we also want to investigate the relative size of the static and the dynamic multiplier. Similar to Section 2.4, we peg q_{t+s} artificially at q^* in order to simulate the static multiplier effect. We have similar to Equation (12):

$$\hat{q}_t\Big|_{q_{t+1}=q^*} = \frac{R-1}{R}\frac{1}{2\eta}\left(\hat{K}_{1t}+\hat{K}_{2t}\right)\Big|_{q_{t+1}=q^*}. \tag{27}$$

Substituting this expression for the land price in Equation (25) leads to two equations for $i = 1, 2$:

$$\hat{K}_{it}\Big|_{q_{t+1}=q^*} + \frac{\hat{K}_{1t}+\hat{K}_{2t}}{2\eta}\left(1+\hat{K}_{it}\right)\Bigg|_{q_{t+1}=q^*} = \left(\delta_{i1}+(-1)^i\frac{\epsilon}{2}\right)\Delta + \frac{1}{2\eta}\left(\hat{K}_{1t}+\hat{K}_{2t}\right)\Big|_{q_{t+1}=q^*},$$

which results into

$$\hat{K}_{1t}\Big|_{q_{t+1}=q^*} = \frac{(2-\epsilon)\eta}{2}\left(\sqrt{\frac{2\Delta}{\eta}+1}-1\right) \quad \text{und} \quad \hat{K}_{2t}\Big|_{q_{t+1}=q^*} = \frac{\epsilon\eta}{2}\left(\sqrt{\frac{2\Delta}{\eta}+1}-1\right). \tag{28}$$

The static multiplier effect for $\hat{K}_{it}$ does, unlike for $\hat{q}_t$, not depend on R. Further, we find that the static effect of the unshocked sector $i = 2$ is direct proportional to the coupling ϵ. As an exemplification, we present in Table 5 the static as well as dynamic multiplier effect on the landholding for different sizes of Δ.

Table 5: Static versus dynamic multiplier effect for the landholding in the extended model, exact solution

This table reports the full instantaneous response, the part due to the static multiplier and the part due to the dynamic multiplier, based on the exact approach. The values for the full response were taken from Table 4 for $R = 1.01$. η and ϵ were chosen 0.1 and 0.5, respectively.

	$\hat{K}_{1t}$			$\hat{K}_{2t}$		
Δ	0.01	0.03	0.05	0.01	0.03	0.05
full response	0.0239	0.0436	0.0578	0.0198	0.0328	0.0411
static multiplier	0.0072	0.0199	0.0311	0.0024	0.0066	0.0104
dynamic multiplier	0.0167	0.0237	0.0267	0.0174	0.0262	0.0307

The static and dynamic multiplier effect are of similar size. This is contrary to the conclusion we would have achieved if the system had been solved via linearization as in Kiyotaki and Moore (1997). This is a similar result as found for the basic model.

4. The Full model

4.1. The Characteristics of the Full Model

The previous analysis did not provide an opportunity to study cycles. Therefore, Kiyotaki and Moore (1997) extended their basic model to make it more realistic. Reproducible capital (trees) is introduced, which deprecates and is specific to the farmer, hence cannot be posted as collateral. Planting fruits can now be seen as an investment, because trees grow out of fruits and later yield fruits. Only a fraction of farmers has an opportunity to invest (plant fruits on uncultivated land), others repay their debt. Therefore, not all framers borrow to the credit limits. This results in an uncoupling of aggregate borrowing and landholding and leads to cycles.

There exists a proportion λ of the land, hold by a farmer, on which trees grow (cultivated land). This landholding at $t-1$ produce $\lambda a k_{t-1}$ tradable fruits and $\lambda c k_{t-1}$ non-tradable fruits at period t. The remaining proportion $(1-\lambda)$ of land has no trees or the trees died. This part is called uncultivated land. Uncultivated land can be bought by gatherers or be recultivated by farmers (by planting fruits which later will grow into trees). In order to increase production, the farmer must increase cultivated landholding. If we assume, that the farmer owns k_{t-1} land of which λk_{t-1} is cultivated, then in order to increase cultivated landholding to k_t the farmer must acquire $k_t - k_{t-1}$ land and plant $\phi(k_t - \lambda k_{t-1})$ fruits. This investment opportunity arises with probability π. The landholding of a farmer who cannot invest is constrained by $k_t < \lambda k_{t-1}$, ie he sells land to gatherers or other farmers. It is assumed that the tradable output is enough to replant *A5:* $a > (1-\lambda)\phi$ and that the probability for investment is not too small *A6:* $\pi > 1 - 1/R$.

The farmers' flow of fund constraint in the full model contains in addition to Equation (1) the 'investment in tree' term $\phi(k_t - \lambda k_{t-1})$:

$$\overbrace{q_t(k_t - k_{t-1}) + \phi(k_t - \lambda k_{t-1}) + R b_{t-1} + \underbrace{x_t - c k_{t-1}}_{\text{consumption above automatic}}}^{\text{cash out}} = \overbrace{a k_{t-1} + b_t}^{\text{cash in}} . \tag{29}$$

Within the steady state the farmer who can invest will, as in the basic model, follow a maximum investment, maximum borrowing and minimum consumption strategy, ie $R b_t = q_{t+1} k_t$ and $x_t = c k_{t-1}$. Following Equation (29) the landholding of a farmer who can invest is:

$$k_t = \frac{1}{\phi + q_t - \frac{1}{R} q_{t+1}} \left[(a + q_t + \lambda\phi) k_{t-1} - R b_{t-1} \right] , \tag{30}$$

where the farmer who cannot invest will not divest, ie

$$k_t = \lambda k_{t-1} \tag{31}$$

and consume also only the bruised fruits. Hence, the farmer who cannot invest will use the proceeds from land sale (the part which is uncultivated) $q_t(1-\lambda)k_{t-1}$ and his tradable output $a k_{t-1}$ to pay off part of his debt.

In other words, the full model has extended the basic model in two ways: Firstly, heterogeneity was introduced via the investment opportunity π and secondly, trees were introduced as an additional investment opportunity. For more details on the characteristics of the full model we refer the reader to Kiyotaki and Moore (1997).

4.2. The Equations of Motion of the Full Model

The structure of Equations (30) and (31) in k_t and b_t allows to aggregate over all farmers and write for the landholding:[10]

$$K_t = \overbrace{(1-\pi)\lambda K_{t-1}}^{\text{farmers who cannot invest}} + \overbrace{\pi \frac{1}{\phi + q_t - \frac{1}{R} q_{t+1}} \left[(a + q_t + \lambda\phi) K_{t-1} - R B_{t-1} \right]}^{\text{farmers who can invest}} . \tag{32}$$

Since the farmer consumes only the bruised fruits, $x_t = c k_{t-1}$, we can derive from Equation (29) the aggregate flow of funds constraint:

$$B_t = R B_{t-1} + q_t (K_t - K_{t-1}) + \phi (K_t - \lambda K_{t-1}) - a K_{t-1} . \tag{33}$$

The steady state solution of the full model is presented in Appendix C and is in full agreement with Kiyotaki and Moore (1997). We concentrate on the dynamics of the full model. Unlike in the basic model, we have no strong coupling between debt and landholding, therefore we have a system of EOM with the three variables: Equations (32), (33) and the equation for the land price:

$$q_t = \sum_{s=0}^{\infty} R^{-s} u(K_{t+s}) . \tag{34}$$

In order to solve the system of equations and run simulations we set again: $u(K) = K - \nu$ with $\nu = 9u^*$. We assume that the system is in the steady state at $t-1$ and a small temporary shock is applied to the system at t. It is convenient to rewrite the EOM (32) to (34) in terms of deviations from the steady state. At date t we find:

$$\begin{aligned}
\hat{q}_t &= \frac{K^*}{q^*} \sum_{l=0}^{\infty} R^{-l} \hat{K}_{t+l} &(35)\\
\hat{K}_t &= \frac{\pi}{\phi + K^* \hat{K}_t + u^*} \left(q^* \hat{q}_t + \phi + q^* + \Delta a - \frac{B^*}{K^*} \right) - 1 + \lambda - \pi\lambda &(36)\\
\hat{B}_t &= q^* (\hat{q}_t + 1) \frac{K^*}{B^*} \hat{K}_t + \phi \frac{K^*}{B^*} \hat{K}_t - \Delta a \frac{K^*}{B^*} . &(37)
\end{aligned}$$

At date $t+s$ with $s \geq 1$ the EOM take the form:

$$\begin{aligned}
\hat{q}_{t+s} &= R \hat{q}_{t+s-1} - R \frac{K^*}{q^*} \hat{K}_{t+s-1} &(38)\\
\hat{K}_{t+s} &= \frac{(\lambda - \lambda\pi - 1) K^*}{\phi + u^* + K^* \hat{K}_{t+s}} \hat{K}_{t+s} + (1-\pi) \lambda \hat{K}_{t+s-1} &(39)\\
&\quad + \frac{\pi}{\phi + u^* + K^* \hat{K}_{t+s}} \left[(a + \hat{q}_{t+s} q^* + q^* + \lambda\phi) \hat{K}_{t+s-1} - R \frac{B^*}{K^*} \hat{B}_{t+s-1} + q^* \hat{q}_{t+s} \right] \\
\hat{B}_{t+s} &= R \hat{B}_{t+s-1} + \frac{K^*}{B^*} \left[q^* (\hat{q}_{t+s} + 1) + \phi \right] \hat{K}_{t+s} - \frac{K^*}{B^*} \left[q^* (\hat{q}_{t+s} + 1) + \phi\lambda + a \right] \hat{K}_{t+s-1}, &(40)
\end{aligned}$$

where we have used Equation (5) to express $\hat{q}_{t+s}$.

[10] See Kiyotaki and Moore (1997) for details.

4.3. The response to a small temporary shock

Equations (35) to (40) are non-linear. Kiyotaki and Moore (1997) linearized the system, which is now due to the complexity of the model a rather non-trivial task. We will initially use their findings (see Equations (C.7) to (C.6) in Appendix C), to get a quantitative understanding of the instantaneous shock response. We choose π, the probability of an investment opportunity, 0.1, 0.5 and 1. We vary the interest rate R between 1.00 and 1.05. Further, as an exemplification, the small temporary shock to the economy is chosen 0.01. The parameter λ, the fraction of trees that do not die, is identical to 0.975. The coefficient ϕ of the 'investment in tree' term is set to 20. As in the previous sections, the intercept in $u(K) = K - 9u^*$ is chosen such that $\eta = 0.1$. Further, we have normalized $a = 1$. The results of the instantaneous changes to the landholding, borrowing and land price are presented in Table 6. Analyzing Equations (C.6) to (C.7) we see: for $\phi = 0$ (no trees) and $\pi = 1$ (all farmers can invest) the linearized results are, as expected, identical to the linearized results of the basic model, hence, as discussed, incorrect. On the other hand, for ϕ reasonably larger than 0 and π well below 1, a linearization is feasible, because inspecting the results in Table 6 (except for the first row) clearly reveals that a small shock produces only small instantaneous deviations from the steady state. Clearly, a linearization of the EOM of the full model produces valid results for some parameters. However, the linearized solution shows again a singularity for $R = 1$, ie for a zero interest rate environment (first row in Table 6), independent of the shock size. This obviously shows the non-economic behavior of the linear solution.

Table 6: Instantaneous response to a temporary shock Δ in the full model, linearized solution

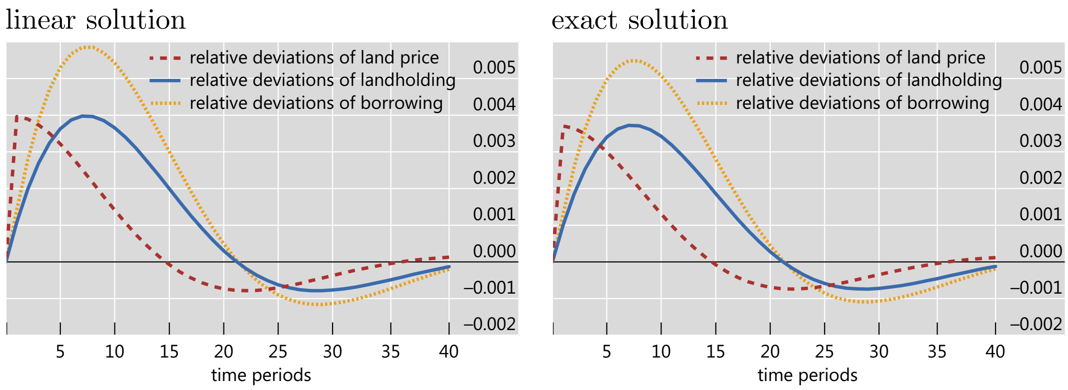
The table reports the relative deviations of the land price, landholding and borrowing from the steady state at time t for a variety of different investment opportunities and interest rates, based on the linearized solution. η, λ, ϕ and a were set to 0.1, 0.975, 20 and 1, respectively. Δ was chosen to be 1%.

	Changes of land price			Changes of landholding			Changes of borrowing		
π	0.1	0.5	1.0	0.1	0.5	1.0	0.1	0.5	1.0
$R = 1.00$	0.0040	0.0048	0.0049	∞	∞	∞	∞	∞	∞
$R = 1.01$	0.0040	0.0048	0.0049	0.0011	0.0055	0.0101	0.0014	0.0076	0.0140
$R = 1.03$	0.0040	0.0048	0.0049	0.0005	0.0020	0.0037	0.0005	0.0040	0.0076
$R = 1.05$	0.0039	0.0048	0.0049	0.0003	0.0013	0.0024	0.0002	0.0032	0.0063

Further, we investigate how the impulse response from a small temporary shock evolves over time. We continue to use the same parameters as in Kiyotaki and Moore (1997) and in previous simulations (see for example Table 6). We present the results of the simulation in Figure 4. The left-hand panel shows the results achieved via linearization of the EOM. The right-hand panel shows the exact solution. From Table 6 it was already evident that a linearization of the EOM for the full model and the chosen parameters is a valid approximation. Hence, the left- and right-hand panel of Figure 4 are almost identical. In fact, the Figure 3 on page 238 in Kiyotaki and Moore (1997) presents the solution of the non-linear model and hence should be compared to the right-hand panel in Figure 4.[11]

Figure 4: Evolution of shock responses in the full model

The figure illustrates the time evolution of relative deviations of the land price, landholding and borrowing from the steady state. The left-hand panel presents the solution achieved via linearization of the EOM, while the right-hand panel shows the solution achieved by including also higher order terms in the EOM. The results are very similar for the chosen parameters: $\pi = 0.1$, $R = 1.01$, $\Delta = 0.01$, $\lambda = 0.975$, $\phi = 20$ and $\eta = 0.1$. Further, we normalized $a = 1$.



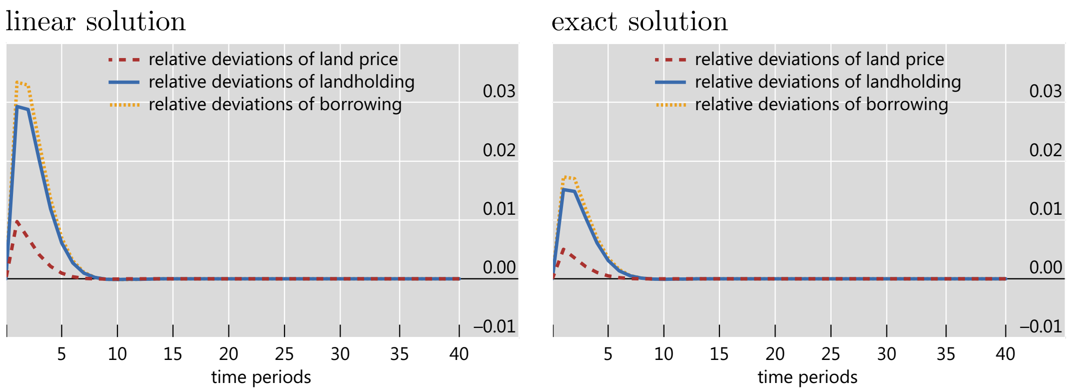
The small temporary shock to production of 1% results in a below 0.6% deviation of borrowing from the steady state and a below 0.4% deviation of the landholding and land price. While the land price reaches its maximum immediately, borrowing and landholding peak around period 7. Overall, the deviations are smaller than the shock and we cannot report amplification. The system, however, oscillates persistently around the steady state.

The full model is of rather complex nature and hence a linearization is not always an appropriate way to solve the model. As discussed, when π gets close to 1 and ϕ gets close to 0 (basic model) a linearization renders incorrect results. In Figure 5, we present the solution for a parameter set closer to the basic model: $\pi = 0.6$ and $\phi = 10$, while all other parameters are kept the same. In the left hand panel, we solved the model using a linear approximation. The right-hand panel presents the exact solution. The first observation is that the boom-bust dynamic has disappeared. Now, a large proportion of farmers have in each period an investment opportunity, which again leads to a strong coupling between landholding and borrowing. Further, reducing ϕ has lowered the impact of the 'investment in tree' term in Equation (29). Therefore, the dynamic is similar to the basic model. The second observation relates to the visible and significant difference of the deviations from the steady state between the left-hand panel and the right-hand panel in Figure 5.

[11] Kiyotaki and Moore (1997) plotted x_t/x^*, while we plotted $(x_t - x^*)/x^*$ with x_t: q_t, K_t or B_t.

Figure 5: Evolution of shock response in the full model

The figure illustrates the time evolution of the model's shock response. π and Φ were chosen 0.6 and 10, respectively. For further information we refer to Figure 4.

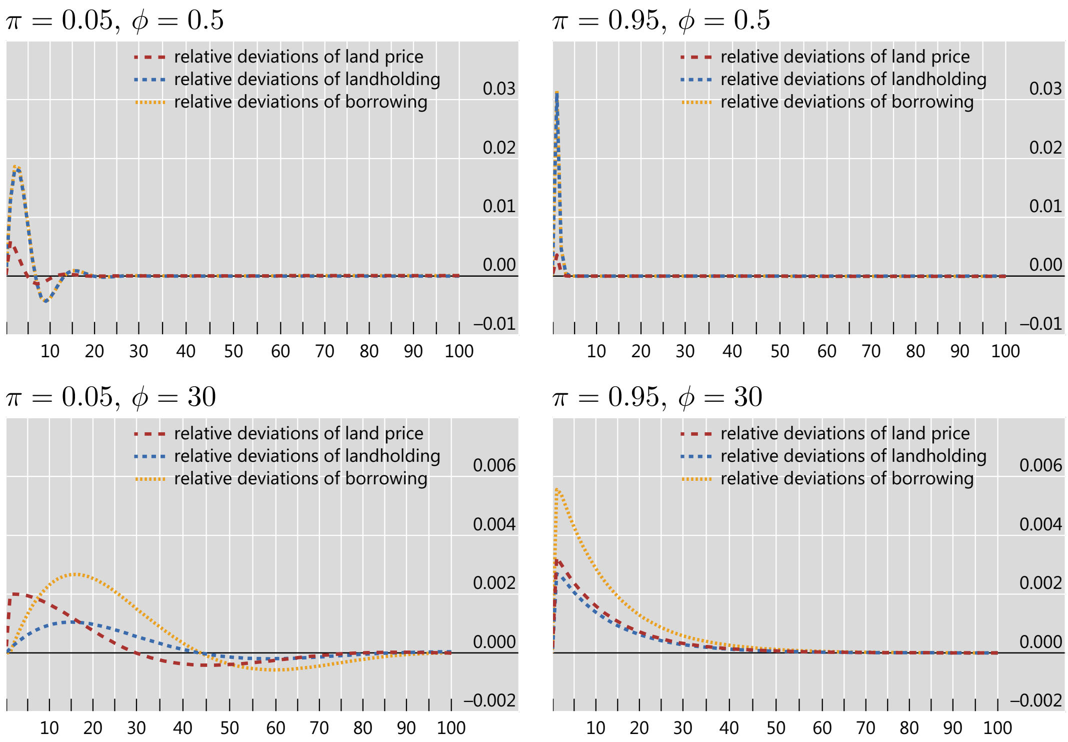


Reducing the size of the 'investment in tree' term in Equation (29) by lowering ϕ reduces the persistence, ie the shock gets quickly reabsorbed (compare Figure 4 with Figure 5). We can conclude that even though the full model contains a rich dynamic it does not show amplification, and persistence is present only for some choices of parameter values. In order to see the behavior of Kiyotaki-Moore's full model at a glance, we present four extreme cases in terms of the parameters π (farmers' heterogeneity) and ϕ (investment opportunity in a non-collateralizable asset) in Figure 6.

From Figure 6, we see that for a high probability of investment, ie π close to 1, we have no boom and bust dynamics. For small values of π landholding and borrowing is decoupled. Farmers do not borrow to their credit limits and any post-shock scenario contains a cyclic behavior. For large values of ϕ, we see more persistence, however, the deviations from the steady state are rather small. A large ϕ increases the relevance of the 'investment in tree' term in the EOM and hence reduces the farmer's leverage, which quite intuitively leads to less severe booms and busts. At the same value of ϕ we find generally larger deviations from the steady state for large values of π, because again for larger values of π the farmer borrows more.

Figure 6: Time evolution of shock responses for four extreme cases of π and ϕ, exact solution

The figure illustrates the evolution of relative deviations from the equilibrium due to a 1% shock. We varied π and ϕ, but kept the other parameters fixed: $\lambda = 0.975$, $\eta = 0.1$, $R = 1.01$ and $a = 1$. The only situation where we can report amplification is the upper right panel, which represents a case close to the unrealistic basic model, where close to all farmers have investment opportunities ($\pi \approx 1$) and there is practically no reproducible capital, ie trees ($\phi \approx 0$).



4.4. Static versus dynamic multiplier effect

Finally, we analyze the dynamic multiplier effect. Again, we artificially peg q_{t+1} at q^*, hence set $\hat{q}_{t+1} = 0$. Following the same arguments as in Section 2.4 Equations (36) and (37) simplify to:

$$\hat{K}_t\Big|_{q_{t+1}=q^*} = \frac{\pi}{\phi + K^*\hat{K}_t\Big|_{q_{t+1}=q^*} + u^*}\left(K^*\,\hat{K}_t\Big|_{q_{t+1}=q^*} + \phi + q^* + \Delta a - \frac{B^*}{K^*}\right) - 1 + \lambda - \pi\lambda \tag{41}$$

$$\hat{B}_t\Big|_{q_{t+1}=q^*} = \left(K^*\,\hat{K}_t\Big|_{q_{t+1}=q^*} + q^*\right)\frac{K^*}{B^*}\,\hat{K}_t\Big|_{q_{t+1}=q^*} + \phi\frac{K^*}{B^*}\,\hat{K}_t\Big|_{q_{t+1}=q^*} - \Delta a\frac{K^*}{B^*}\,. \tag{42}$$

Table 7: The full instantaneous response of $\hat{K}_t$ and its two components: the static and dynamic multiplier, exact solution

The table reports for different ϕ and π the full instantaneous response to a temporary shock $\Delta = 0.01$, using the exact solution, and its two components: the static and dynamic multiplier. R, η and λ were chosen 1.01, 0.1 and 0.975, respectively. The boldfaced values correspond to the basic model.

	Full instantaneous response			Static response			Dynamic response		
π	0.2	0.6	1.0	0.2	0.6	1.0	0.2	0.6	1.0
$\phi = 0$	0.0256	0.0305	**0.0310**	0.0016	0.0052	**0.0092**	0.0240	0.0253	**0.0218**
$\phi = 10$	0.0065	0.0152	0.0204	0.0002	0.0006	0.0009	0.0063	0.0146	0.0195
$\phi = 20$	0.0021	0.0058	0.0090	0.0001	0.0003	0.0005	0.0020	0.0055	0.0085

As discussed, the numerical values of the instantaneous response to a small temporary, shock in the full model using the exact (non-linearized) solution are small, also for $\phi = 0$ and $\pi = 1$ (basic model). The static response is negligible, except for small values of ϕ. The dynamic multiplier effect dominates relative to the static effect, especially for larger ϕ values. Hence, the value of ϕ determines the persistence. Despite being dominant, the dynamic multiplier effect is small in absolute terms.

5. Conclusion

We analyzed the model developed by Kiyotaki and Moore (1997) and could prove that a small temporary shock to the production in a credit constrained economy does not lead to large deviations from the steady state. In fact, the large deviations reported by Kiyotaki and Moore (1997) are due to an invalid linearization of the EOM. Taking only linear terms of the Taylor expansion in $\hat{K}_t$ in the EOM, spuriously leads to a solution which generates large deviations of the landholding from the steady state ($\hat{K}_t$ in the order of 1) even for small shocks. This is a clear inconsistency, as a linearization in $\hat{K}_t$ requires small values of $\hat{K}_t$ so that higher order terms are negligible. For $R = 1$ the linearized solution provides even a singularity, which is non-economical and not present in the exact solution. Further, the dynamic multiplier effect is only dominant compared to the within period effect in the full model, but in absolute values still rather small. In the basic model, we find that the inter-temporal multiplier and the static multiplier effect are of similar size. The same results apply to the extended model with two sectors. In the extended model, we find spillovers between the two sectors, however, the large spillovers reported by Kiyotaki and Moore (1997) are spurious again due to the incorrect linearization. A small shock to one sector generates small deviations to the land price and landholding in both sectors if the exact EOM is solved, whereby the deviations in the undisturbed sector are smaller.

The more realistic full model developed by Kiyotaki and Moore (1997) has been extended by two features: heterogeneity of farmers, described by $\pi \in [0, 1]$ (probability of an investment opportunity) and the introduction of reproducible capital, described by $\phi \geq 0$ (size of the 'investment in tree' term). For some values of π and ϕ, as well as R reasonably well away from 1, the proposed linearization by Kiyotaki and Moore (1997) leads to valid results, because the resulting deviations of landholding are small. For π close to 1 and ϕ close to zero, the linearization produces incorrect results. The linearized solution of the full model provided in Kiyotaki and Moore (1997) shows again a singular behavior for $R = 1$ (zero interest rate environment). Solving the EOM exactly leads to consistent results without pole positions. We could prove that the full model does not lead to amplification, ie a small shock leads to small deviations from the steady state. As reported by Kiyotaki and Moore (1997), the full model contains a rich dynamics, eg boom and bust cycles for small π and large ϕ. Large values of ϕ lead to larger persistence but the maximal deviation from the equilibrium, ie the severity of the boom or bust, is smaller. The dynamic multiplier effect is dominant compared to the static multiplier effect, except for small ϕ. However, the overall size of the dynamic multiplier is small.

Credit constraints do not lead to an amplification of shocks or large spillovers. In fact, for small π, the situation where farmers stay below their credit constraints, the amplitude of the boom or bust is smaller. Regulators and central banks should always be worried if balance sheets of financial institutions are expanding unsustainably. Additional investment opportunities (modeled via ϕ) reduce the strength of the booms and busts even more, however introduce persistence.

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Appendix A. Basic Model: Steady State Solution and Linear Dynamics

We present a brief derivation of the perfect foresight equilibrium. We assume a fixed land supply: $K_t + m\tilde{k}_t = \overline{K} =$ constant. Hence, it follows from Equation (5)

$$u_t = q_t - \frac{1}{R}q_{t+1} = u(K_t) = \frac{1}{R}G'\left(\frac{1}{m}(\overline{K} - K_t)\right) \tag{A.1}$$

and therefore, for the steady state

$$u^* = \frac{R-1}{R}q^* \qquad \text{and} \qquad u^* = \frac{1}{R}G'\left(\frac{1}{m}(\overline{K} - K^*)\right). \tag{A.2}$$

The farmers' aggregate EOM (4) reduce in equilibrium to

$$u^* = a \qquad \text{and} \qquad B^* = \frac{1}{R}q^*K^* = \frac{a}{R-1}K^*. \tag{A.3}$$

The interpretation of these steady state equations is straight forward: the farmers tradable output aK^* is identical to the interest repayment $(R-1)B^*$ and the down payment per unit of land u^* is identical to the tradable output per unit of land a. The land price in equilibrium are linked to fundamentals: $q^* = R/(R-1)a$.

The aggregate farmers' marginal product is $F'(k_t) = a + c$ and the gatherers' $G'(\tilde{k}_t) = Ru_t$. In an economy without credit constraints, the land usage would be K_0, the intercept point of F' and G' (see Figure A.7). Hence, the competitive land price for an economy without credit constraints in equilibrium is given by:

$$F'(k_t) = G'(\tilde{k}_t) = a + c = Ru_0 = R(q_0 - \frac{1}{R}q_0) = q_0(R-1) \to q_0 = \frac{a+c}{R-1}. \tag{A.4}$$

In a credit constrained economy we know from Equations (A.2) and (A.3) that

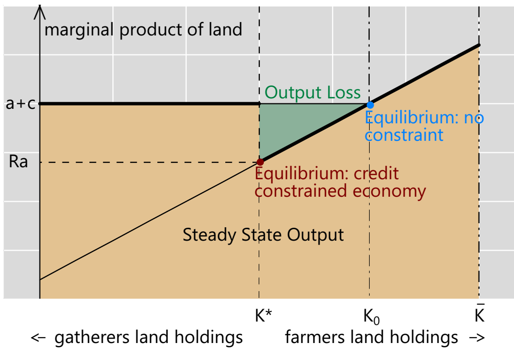
$$G'(\tilde{k}_t) = Ru^* = R(q^* - \frac{1}{R}q^*) = q^*(R-1) = Ra \to q^* = \frac{Ra}{R-1}. \tag{A.5}$$

Due to assumptions *A1* and *A2* we know $a + c > a/\beta > a/\tilde{\beta} = Ra$, ie $q^* < q_0$. Hence, in equilibrium the land usage K^* in the credit constrained economy is less than the land usage K_0 in an economy without credit constraints (see Figure A.7).

Kiyotaki and Moore (1997) aimed to study the effect of a small temporary shock Δ to production. They concluded therefore that a linearization of the EOM (6) and (7) is

Figure A.7: Graphical interpretation of the steady state equilibrium

The figure illustrates the steady state for a credit constrained economy as well as for an economy without credit constraints. The green area represents the output loss due to credit constraints. Within this model, the land usage in the credit constrained economy K^* is less than the land usage K_0 in an economy without credit constraints. Reproduced from Kiyotaki and Moore (1997).



a valid method. Linearizing Equations (9) to (11) leads to the results in Kiyotaki and Moore (1997):

$$\left(1+\frac{1}{\eta}\right)\hat{K}_t = \Delta + \frac{R}{R-1}\hat{q}_t \qquad \text{at } t\,, \tag{A.6}$$

$$\left(1+\frac{1}{\eta}\right)\hat{K}_{t+s} = \hat{K}_{t+s-1} \qquad \text{at } t+s \text{ with } s \geq 1\,, \tag{A.7}$$

$$\hat{q}_t = \frac{R-1}{R}\frac{1}{\eta}\sum_{s=0}^{\infty} R^{-s}\hat{K}_{t+s} = \frac{R-1}{\eta}\frac{(1+\eta)}{R(1+\eta)-\eta}\hat{K}_t\,, \tag{A.8}$$

where Equation (A.8) has been derived by recursively substituting Equation (A.7) into the infinite sum of Equation (A.8).

From Equations (A.6) and (A.8) we can compute the instantaneous changes in land-holding and land price due to a small shock Δ. Solving for $\hat{q}_t$ and $\hat{K}_t$ is straight forward:

$$\hat{q}_t = \frac{1}{\eta}\Delta \qquad \text{and} \qquad \hat{K}_t = \frac{1}{1+\eta}\left(\eta + \frac{R}{R-1}\right)\Delta\,. \tag{A.9}$$

Assuming economically reasonable values[12] of η, the jump in the land price $\hat{q}_t$ is of the same order of magnitude as the shock Δ. However, the landholding can deviate dramatically from the steady state even for small shocks. The reason behind the latter observation is the leverage factor $R/(R-1)$. In other words, a small shock can produce large deviations of K_t from the steady state.

[12] Using the relation $m\tilde{k} = \overline{K} - K$ and the assumption $G'' < 0$, we can conclude from: $\frac{1}{\eta} = \left.\frac{d\ln u(K)}{d\ln K}\right|_{K=K^*} = \left.\frac{d\ln(G'(\tilde{k})/R)}{d\ln K}\right|_{K=K^*} = \left.\frac{d\ln\tilde{k}}{d\ln K}\frac{\ln G'(\tilde{k})}{d\ln\tilde{k}}\right|_{\tilde{k}=(\overline{K}-K^*)/m} = \left.\frac{-K^*}{\overline{K}-K^*}\frac{\ln G'(\tilde{k})}{d\ln\tilde{k}}\right|_{\tilde{k}=(\overline{K}-K^*)/m}$, that $\eta > 0$. For a reasonable total amount of land $\overline{K}$, steady state land demand K^* and functional form of the gatherers' production function around the steady state, we can infer that η is in the order of 1.

Appendix B. Extended Model: Steady State Solution and Linear Dynamics

In order to find a simple analytic solution for the steady state we assume, like Kiyotaki and Moore (1997), symmetry between both sectors, ie equal productivity as well as land- and debt-holdings: $a_1 = a_2 = a$, $K_1^* = K_2^* = K^*/2$ and $B_1^* = B_2^* = B^*/2$ in equilibrium. With this simplification, it follows directly from Equation (17):

$$K^* = \frac{1}{u^*} a p_{it}^* K^* \qquad \text{and} \qquad B^* = \frac{1}{R} q^* K^* \,. \tag{B.1}$$

Further, from Equation (16) we find for a symmetric model in equilibrium

$$\begin{aligned} p_{it}^* &= (a_i K_i^*)^{-\epsilon} \left[(a_1 K_1^*)^{1-\epsilon} + (a_2 K_2^*)^{1-\epsilon} \right]^{\epsilon/(1-\epsilon)} \\ &= (a_i K_i^*)^{-\epsilon} \left[2 (a_i K_i^*)^{1-\epsilon} \right]^{\epsilon/(1-\epsilon)} = 2^{\epsilon/(1-\epsilon)} \,. \end{aligned} \tag{B.2}$$

It follows therefore from Equations (18) and (B.1):

$$u^* = 2^{\epsilon/(1-\epsilon)} a = \frac{R-1}{R} q^* \qquad \text{and} \qquad B^* = \frac{1}{R} q^* K^* = \frac{2^{\epsilon/(1-\epsilon)} a}{R-1} K^* \,, \tag{B.3}$$

which is identical to the basic model, if we set $2^{\epsilon/(1-\epsilon)} a = a$.

Kiyotaki and Moore (1997) analyzed the dynamics of the two sector model by linearizing the EOM. We use Equations (25) and (26), and neglect all quadratic terms:

$$\hat{K}_{it} + \frac{\hat{K}_{1t} + \hat{K}_{2t}}{2\eta} \approx \left(\delta_{i1} + (-1)^i \frac{\epsilon}{2} \right) \Delta + \frac{R \hat{q}_t}{R-1} \qquad \text{at } t \,, \tag{B.4}$$

$$\hat{K}_{it+s} + \frac{\hat{K}_{1t+s} + \hat{K}_{2t+s}}{2\eta} \approx \hat{K}_{it+s-1} + (-1)^i \frac{\epsilon}{2} (\hat{K}_{1t+s-1} - \hat{K}_{2t+s-1}) \text{ at } t+s \text{ for } s \geq 1 \,, \tag{B.5}$$

where in the last equation the linearized expression of $p_{it+s}/2^{\epsilon/(1-\epsilon)}$ has been inserted:

$$\frac{p_{it+s}}{2^{\epsilon/(1-\epsilon)}} = 1 + (-1)^i \frac{\epsilon}{2} (\hat{K}_{1t+s-1} - \hat{K}_{2t+s-1}) \,. \tag{B.6}$$

The Equation (A.8) needs to be adjusted to two sectors. As explained in Section 3.3, this results in an additional factor 1/2:

$$\hat{q}_t = \frac{R-1}{2\eta} \frac{(1+\eta)}{R(1+\eta) - \eta} S_t \,. \tag{B.7}$$

In order to compute the instantaneous shock response we take the EOM (B.4) and replace $\hat{q}_t$. Subtracting and adding $\hat{K}_{1t}$ and $\hat{K}_{2t}$ leads to:

$$\hat{K}_{1t} - \hat{K}_{2t} = \Delta - \epsilon \Delta \qquad \text{and} \qquad \hat{K}_{1t} + \hat{K}_{2t} = \frac{1}{1+\eta} \left[\eta + \frac{R}{R-1} \right] \Delta \,, \tag{B.8}$$

which can now be solved for $\hat{K}_{1t}$ and $\hat{K}_{2t}$. The instantaneous response of the landholding for both sectors to a small temporary shock are:

$$\hat{K}_{1t} = \left[1 + \frac{1}{2(\eta+1)(R-1)} - \frac{1}{2}\epsilon\right]\Delta \quad \text{and} \quad \hat{K}_{2t} = \left[\frac{1}{2(\eta+1)(R-1)} + \frac{1}{2}\epsilon\right]\Delta\,. \tag{B.9}$$

The instantaneous response to a shock consists of three parts: The direct impact of the shock to $\hat{K}_{1t}$, the indirect impact due to changes in the land price, and the term proportional to ϵ representing the demand linkage.

This result is in agreement[13] with Kiyotaki and Moore (1997). Again, a small shock can produce large deviations from the equilibrium due to the leverage factor $1/(R-1)$. Further, both sectors respond similarly strongly to the shock, despite the fact, that the shock is applied to sector 1. The reason for these findings lies again in the linearization. The large spillover disappears as soon as the EOM are solved exactly.

In order to get the time evolution of the shock response, we need to use the EOM (B.5). Employing linear algebra yields the simple expression for the evolution of the landholding for $s \geq 1$:

$$\begin{pmatrix} \hat{K}_{1t+s} \\ \hat{K}_{2t+s} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2(1+\eta)} - \frac{\epsilon}{2} & -\frac{1}{2(1+\eta)} + \frac{\epsilon}{2} \\ -\frac{1}{2(1+\eta)} + \frac{\epsilon}{2} & 1 - \frac{1}{2(1+\eta)} - \frac{\epsilon}{2} \end{pmatrix} \begin{pmatrix} \hat{K}_{1t+s-1} \\ \hat{K}_{2t+s-1} \end{pmatrix}, \tag{B.10}$$

which is in agreement with Kiyotaki and Moore (1997).

Appendix C. The Full Model: Steady State Solution and Linear Dynamics

We reproduce the perfect foresight equilibrium, which is consistent with Kiyotaki and Moore (1995) and Kiyotaki and Moore (1997). Equation (33) yields

$$B^* = \frac{1}{1-R}(\phi - \phi\lambda - a)K^*\,, \tag{C.1}$$

which together with Equation (32) leads to the following expression for q^*:

$$\begin{aligned} 1 &= (1-\pi)\lambda + \pi\frac{1}{\phi + q^* - \frac{1}{R}q^*}\left[(a + q^* + \lambda\phi) - \frac{R}{R-1}(a - \phi + \lambda\phi)\right], \\ \frac{R-1}{R}q^* &= \frac{\pi a - (1-\lambda)(1-R+\pi R)\phi}{\lambda\pi + (1-\lambda)(1-R+\pi R)}\,. \end{aligned} \tag{C.2}$$

The land market clearing condition remains unchanged, hence

$$u^* = \frac{R-1}{R}q^* \qquad \text{and} \qquad \frac{1}{R}G'\left(\frac{1}{m}(\bar{K} - K^*)\right) = u^*\,. \tag{C.3}$$

[13] Adding Equations (B.9) gives the same shock response as in Equation (A.9) for the basic model. We apply the shock only to sector 1 and would therefore expect only half of the shock response for our symmetrical system, consistent with the statement above Equation (35a) in Kiyotaki and Moore (1997). However, we have to remember that $\hat{K}_{1t} + \hat{K}_{2t}$ is twice the deviation from the equilibrium ($\hat{K}_t = (K_{1t} + K_{2t} - K_1^* - K_2^*)/(K_1^* + K_2^*) = 1/2\,(\hat{K}_{1t} + \hat{K}_{2t})$). In so far the equations are correct, but the statement above Equation (35a) in Kiyotaki and Moore (1997) contains a typo. Only the proportional change in the land price contains a factor 1/2, not the proportional change in the farmers' combined landholding.

It is worth observing that when we set $\pi = 1$ (remove heterogeneity) and $\phi = 0$ (remove trees) we get Equations (A.3), ie the steady state of the basic model.

Kiyotaki and Moore (1997) have taken the land price $\hat{q}_{t+s}$ as a jump-variable. The linearized solutions represent trajectories on a plane attached to the non-linearized curved manifold at the steady state. The plane is expressed in terms of deviations from the steady state (see Kiyotaki and Moore (1997)):

$$\hat{q}_{t+s} = \frac{\pi\Theta}{u^*}\frac{1}{\eta(1-\lambda+\lambda\pi)}\left((\phi+q^*)\hat{K}_{t+s} - \frac{B^*}{K^*}\hat{B}_{t+s}\right), \tag{C.4}$$

where $\Theta \equiv u^*/(\phi + u^*)$.

Linearizing Equations (36) and (37) as well as using Equation (C.4) for $s = 0$ we find two linear equations for two unknowns and can compute the instantaneous response to a small temporary shock at date t:

$$\hat{K}_t = \frac{\pi\Theta}{u^*}\frac{1}{(R-1)(1-\lambda+\lambda\pi)}\frac{\eta(R-1)(1-\lambda+\lambda\pi)+\pi R\Theta}{\eta+(1-\lambda+\lambda\pi)\Theta}a\Delta\,, \tag{C.5}$$

$$\hat{B}_t = \frac{K^*}{B^*}\left((\phi+q^*)\hat{K}_t - a\Delta\right). \tag{C.6}$$

Again, we recognize the factor $R/(R-1)$, which for small R renders the linearization incorrect. Using the results in Equations (C.5) and (C.6) and inserting them into Equation (C.4) for $s = 0$ we find the instantaneous response of the land price:

$$\hat{q}_t = \frac{\pi\Theta}{u^*}\frac{1}{\eta(1-\lambda+\lambda\pi)}a\Delta\,. \tag{C.7}$$

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