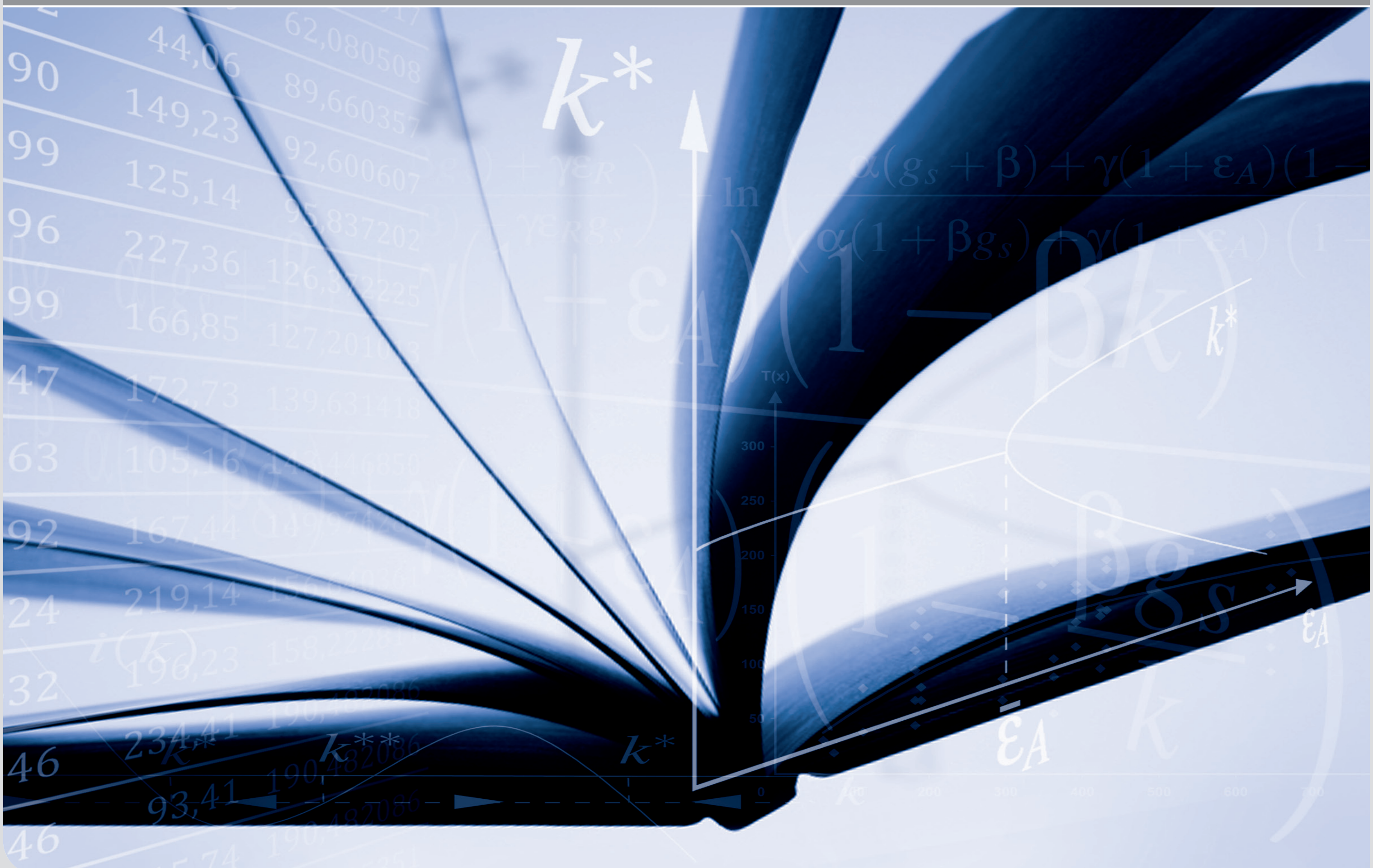
Karlsruher Institut für Technologie

Apportionment in times of digitalization

by Andranik S. Tangian

No. 161 | APRIL 2023

WORKING PAPER SERIES IN ECONOMICS



Impressum

Karlsruher Institut für Technologie (KIT)
Fakultät für Wirtschaftswissenschaften
Institut für Volkswirtschaftslehre (ECON)

Kaiserstraße 12
76131 Karlsruhe

KIT – Die Forschungsuniversität in der Helmholtz-Gemeinschaft

Working Paper Series in Economics
No. 161, April 2023

ISSN 2190-9806

econpapers.wiwi.kit.edu

Institute of Economic Theory and Operations Research
Karlsruhe Institute of Technology

Apportionment in Times of Digitalization[1]

Andranik S. Tangian

Working paper Nr. 161

April 2023

E-mail: andranik.tangian@kit.edu
andranik.tangian@gmail.com
Tel: +49 721 6084 3077

Blücherstraße 17 76185 Karlsruhe Deutschland

[1]Based on the paper at the 60th meeting of the Public Choice Society, Seattle, March 16–18, 2023.

Abstract

We critically discuss the Jefferson/D'Hondt and Webster/Sainte-Laguë methods, which are used to allocate parliament seats to parties in the mixed-member proportional representation systems in Germany, New Zealand, Bolivia, South Africa, South Korea, Scotland and Wales, as well as in the European Parliament. The task is as follows: (1) the parliament must be of a certain size or slightly larger than that, (2) the party factions must include all direct mandate holders elected in constituencies and (3) the faction ratio should reflect, with a certain accuracy, the votes received by the parties across the country. We show that discrete optimization techniques result in better and more accurate apportionments. In addition, we consider adjustment vote weights defined within the optimization approach and show that they can give a general consistent solution to the apportionment problem. All of these are illustrated using the example of the 2021 German Bundestag elections.

Keywords: representative democracy, proportional representation, apportionment, Jefferson/D'Hondt method, Webster/Sainte-Laguë method, optimization, adjustment vote weights.

JEL Classification: D71

Contents

All errors in government and in society are based on philosophic errors, which in turn are derived from errors in natural science.

Nicolas de Condorcet (1743–1794)[2]

1 Introduction

Digitalization, affecting many areas of human activity, includes, among other things, optimization issues. Optimization, as a part of computational philosophy, is essentially opposed to the axiomatic/rule-based or legal regulations. Optimization deals with objectives that are fixed at the starting point, and if they are not achievable completely because of constraints, they are approached. On the contrary, axiomatization deals with yes/no acceptability of possible solutions, which marginalizes those in the 'grey zone'. All of these were the content of the 1930s debates provoked by Gödel's discoveries of axiomatic incompleteness. Thanks to these debates mathematicians turned from axiomatization to computability, which gave rise to the theory of algorithms and finally led to the invention of computers. In a sense, digitalization is in line with these developments, striving for computability and optimization on a local scale, somewhat pushing aside universal rules and general recommendations.

Therefore, while studying apportionment, we do not focus on analyzing its 'axiomatic properties' but go directly to the apportionment optimization and compare the results obtained with those obtained by the existing methods. From this viewpoint, we examine the 19th century Jefferson/D'Hondt and the similar Webster/Sainte-Laguë apportionment methods [D'Hondt method 2023, Sainte-Laguë method 2023] that are currently used to allocate parliament seats to parties in mixed-member proportional representation systems, which are adopted in the European Parliament, Germany, New Zealand, Bolivia, South Africa, South Korea, and some other countries [Mixed-member proportional representation 2023].[3] In particular, the German Bundestag had allocated seats to parties by the D'Hondt method until 2009 and since then — by the Sainte-Laguë method [D'Hondt-Verfahren 2023, Sainte-Laguë-Verfahren 2023].

These two methods aim at 'maximizing proportionality' [Medzihorsky 2019], that is, implicitly refer to optimization. The 'maximizing proportionality' is understood as 'optimally' fitting the integer number of parliament seats to the fractional shares of the parties' electoral votes. Minimizing the fitting errors can be expressed in either absolute or relative terms. For instance, if a party with 50% of the votes receives 49.5% of the seats then the absolute fitting error is equal to $50 - 49.5 = 0.5\%$. The same absolute fitting error holds for a party with 10% of the votes and 9.5% of the parliament seats. However, the relative 'loss of influence' (fitting error relative to the share of votes) for the first party is equal to $0.5/50 = 1/100 = 1\%$ whereas for the second party it is equal to $0.5/10 = 5/100 = 5\%$, meaning that the same absolute fitting error is perceived differently by large and small factions. Therefore, we consider fitting errors as either absolute or relative deviations from the vote shares.

For computations and other purposes, we introduce *adjustment vote weights* of parliament members, similar to power indices in the game theory [Shapley and Shubik 1954, Mazurkiewicz and Mercik 2005, Varela and Prado-Dominguez 2012, Holler and Nurmi 2013]. Obviously, even if the fractional parliamentary party quotas are not respected, the quota-equal voting power of each faction can be restored by adjusting the vote weights of its members. The computational advantage of this device follows from

[2] Condorcet N (1791) Report and Draft. Decree on the General Organization of Public Instruction. In: Courtel Ch, Kintzler C (Eds) (1994) Condorcet. Cinq mémoires sur l'instruction publique [Five briefs on public education]. Flammarion, Paris. http://classiques.uqac.ca/classiques/condorcet/cinq_memoires_instruction/cinq_memoires.html.

[3] [Medzihorsky 2019] enumerates further applications of the D'Hondt method: (1) allocations of parliamentary committee chairs and executive body seats [O'Leary Grofman and Elklit 2005], (2) allocating claims on indivisibilities [Herrero and Martínez 2008], (3) constrained portfolio selection in economics [Baixauli-Soler Alfaro-Cid and Fernandez-Blanco 2011], (4) rounding numbers in statistics [Dorfleitner and Klein 1999], (5) codon usage fitting in biochemistry [Takai 2016], and (6) usage within algorithms in computer science [Kruse et al. 2016, Zhang et al. 2016, Kubota and Aritsugi 2017].

the observation that optimal apportionments (with minimal absolute or relative deviations from the party quotas) can be found by minimizing the range of adjustment vote weights, and vice versa. As for their interpretation, deviations from 'the ideal of one man, one vote' [Balinski and Young 1982] are not uncommon. For instance, the chairperson of a committee with an even number of members may be given 1.5 votes to avoid a tie. In joint-stock companies, the vote power of each shareholder is proportional to his/her percentage of shares [Edelman et al. 2014], etc. The adjustment vote weights do not in the least contradict the established apportionment practices that already include several types of adjustments like faction size rounding, adding leveling (adjustment) seats or tolerating unadjusted seats (= adjusting the accuracy thresholds). As for the implementation, the fraction-valued voting that obviously solves all apportionment problems is easy to implement in voting consoles with electronic deputy cards.

Thus, we formulate and study the apportionment optimization problem in two versions (optimizing either absolute or relative deviations from the parliamentary party quotas), illustrating it by apportioning the German Bundestag in various ways using the 2021 federal elections outcomes. We find that the Jefferson/D'Hondt and the Webster/Sainte-Laguë methods result in the solutions, which from time to time coincide with the ones obtained using one or another version of the optimization model, depending on the parliament size. The resulting apportionments for smaller parliaments lay between the two optimal solutions, alternately approaching one or another as the parliament grows. For larger parliaments, the resulting apportionments start to coincide with one or another optimal solution more frequently, however, in a quite irregular way. We conclude that the Jefferson/D'Hondt and the Webster/Sainte-Laguë methods lack the 'optimization consistency', which in fact could be hardly expected from heuristic methods of the 19th century.

As for the German Bundestag, the adjustment vote weights could make its numerous adjustment seats unnecessary (instead of its 598 nominal seats, the 2021 Bundestag had 736 ones, moreover, the deviation from party quotas were up to 3.3 seats contrary to the prescribed accuracy of 0.5 seat). For instance, the 2021 Bundestag could be reduced to 630 members, as recommended by Norbert Lammer, the 12th President of the Bundestag from 2005 to 2017, which would require the adjustment vote weights ranging from 1 to 1.29, i.e., within 29% only [Roßner 2016]. Thereby, one can reduce the Bundestag on the one hand, and, on the other hand, refine the ratio of party powers, bringing it to the exact ratio of votes cast for the parties, as opposed to the currently tolerated approximate ratio.

Section 2, 'Jefferson/D'Hondt and Webster/Sainte-Laguë apportionment method', explains two similar methods to allocate parliamentary seats to eligible parties.

In Section 3, 'Apportionment as an optimization problem', the task is reduced to two standard mixed-integer linear programming problems: to optimize either absolute or relative apportionment accuracy. The directed search for their solution is simplified by introducing adjustment vote weights of members of parliament, which compensate the apportionment inaccuracies. The desired apportionment is found using two-level lexicographic optimization: when the selected optimization criterion results in several solutions then the other criterion is applied to choose among them.

In Section 4, 'Example: Apportionment of the 2021 German Bundestag', the D'Hondt and Sainte-Laguë methods and the two optimization models are applied to apportion the Bundestag with the actual 736 seats and 630 seats, as proposed by the former Bundestag president Norbert Lammert.

In Section 5, 'The D'Hondt and Sainte-Laguë methods vs optimization', the performance of the heuristic methods of apportionment is compared with that of two optimization models. It is shown that the Jefferson/D'Hondt and Webster/Sainte-Laguë methods lead to solutions that irregularly approach alternative optimal apportionments, lacking however 'optimization consistency' in their performance.

In Section 6, 'Conclusions', the main findings are recapitulated and put into context.

2 Jefferson/D'Hondt and Webster/Sainte-Laguë apportionment methods

The Jefferson/D'Hondt apportionment method was originally proposed in the United States by Thomas Jefferson as early as 1792. In Europe, it is attributed to the Belgian mathematician and lawyer Victor D'Hondt who reinvented it in 1878 [D'Hondt 1882, D'Hondt 1885, D'Hondt method 2023]. This method has numerous mathematical advantages but it is also known for slightly favoring large parties over small ones [Balinski and Young 1979, Lijphart 2003, Pukelsheim 2007, D'Hondt-Verfahren 2023].

The Webster/Sainte-Laguë method strives to complete the same task as the D'Hondt method and is very similar to it, being its minor modification. It is named after the American statesman Daniel Webster, who proposed it in 1832 for proportional allocation of seats in the United States congressional apportionment [Balinski and Young 1982], and the French mathematician André Sainte-Laguë, who independently rediscovered it and studied its properties [Sainte-Laguë 1910, Sainte-Laguë method 2023]. In Europe, both methods are referred to as D'Hondt and Sainte-Laguë methods, respectively, and we call them following the European tradition.

In 1980, the German physicist and electoral expert Hans Schepers, having studied the D'Hondt method used by the German Bundestag, discovered that it disadvantaged smaller parties and suggested an improved version equivalent to the Sainte-Laguë method [Pukelsheim 2002]. At first it was adopted only for certain Bundestag commissions, but since 2009 it has been used to allocate seats both in the German Bundestag and the European Parliament [Sainte-Laguë-Verfahren 2023]. Both the D'Hondt and Sainte-Laguë methods are widely used worldwide, sometimes interchangeably.

The idea of these methods is as follows. The party with the most electoral votes 'purchases' its first parliamentary seat by 'spending' a certain fixed fraction of the total votes it received in the election. At each successive step, the currently 'richest' party acquires a seat, spending a certain fraction of its remaining votes. Thereby, the next seat goes up to the 'highest bidder' — the party with the most votes to spend. In this way, the biggest winners can acquire several seats before a minor party ever gets to make its first 'purchase'. The procedure runs as long as the initial pool of S nominal parliamentary seats is not exhausted.

The only difference between the D'Hondt and the Sainte-Laguë methods is the amount of spending for each purchase. Under the D'Hondt method, the party 'pays' for the first seat an amount that leaves it with only 1/2 of its original number of votes; then for its next seat it pays an amount that leaves it with only 1/3 of its original number of votes, then 1/4, and so on.

Under the Sainte-Laguë method, the party 'pays' for the first seat an amount that leaves it with only 1/3 of its original number of votes; then for its next seat it pays an amount that leaves it with only 1/5 of its original number of votes, then 1/7, and so on. As one can see, the biggest winners 'spend' their votes much faster than under the D'Hondt method, thereby giving way to smaller parties.

Thus, to allocate the next available seat, the algorithm finds the party i with the *largest remainder of votes* in two versions:

$$\text{while } \sum_{i=1}^{n} x_i < S \quad \text{find } i : \max_{i=1,\ldots,n} \begin{cases} \underbrace{\left(\dfrac{v_i}{x_i+1}\right)}_{\substack{i\text{th party's}\\ \text{remainder}\\ \text{of votes}}} & \text{under D'Hondt method} \\[4em] \underbrace{\left(\dfrac{v_i}{2x_i+1}\right)}_{\substack{i\text{th party's}\\ \text{remainder}\\ \text{of votes}}} & \text{under Sainte-Laguë method} \end{cases} \quad \Rightarrow \quad x_i = x_i + 1\ , \tag{1}$$

where

n is the number of parties eligible for parliamentary seats,

$i = 1, \ldots, n$ are 'labels' of the parties eligible for parliamentary seats,

x_i is the number of seats that have already been allocated to party i (initially $x_i = 0$),

S is the total number of parliamentary seats to be allocated, and

v_i is the total number of electoral votes that party i received in the elections.

If the given number of seats S is insufficient to guarantee the required accuracy of proportionality to votes then *leveling seats* (also called *adjustment seats*) are added one-by-one until the required accuracy is achieved, i.e. the actual number of seats is increased up to $S+\Delta$, where Δ is the number of leveling seats. For example, the German Bundestag that has 598 nominal seats is enlarged further to achieve the accuracy of factions' proportionality to party votes to within the 0.5 seat.

The case of minimum seats reserved for certain parties The situation becomes more complicated when some seats are reserved for certain parties — like a number of seats in the German Bundestag are reserved for parties' local representatives. In this case, the allocation of seats takes place in the regular way step-by-step, with the reserved seats being 'redeemed' first. The reserved seats that are 'not redeemed yet' are considered unadjusted but still belonging to the parliament, making its size at the current computation step greater than the number of seats allocated so far through the regular procedure. For example, the German Bundestag is allocated under these conditions: with a number of reserved seats and a toleration of three unadjusted seats.

3 Apportionment as an optimization problem

The algorithm (1) does not appear to have any traces of optimization criteria, being purely heuristic. One can only admire the intuition of its inventors who so implicitly implemented the optimization idea. It comes however into play when we speak of minimizing the parliament size or 'maximizing proportionality' of the apportionment.

To formulate rigorously the respective optimization model, let us suppose that $i=1,\ldots,n$ are labels of n parties that, after elections, are entitled to seats in the parliament with S seats. We define the following vectors:[4]

$\mathbf{v}=(v_1,\ldots,v_n)'$, — the vote vector, where $v_i>0$ are the parties' integer-valued numbers of votes received in the election;

$\mathbf{q}=(q_1,\ldots,q_n)'$ — the quote vector, where $q_i\geq 0$, $\sum_{i=1}^{n} q_i=1$, are the parties' fraction-valued quotas for parliamentary seats; basically the quotas are proportional to votes: $q_i=\dfrac{v_i}{\sum_{i=1}^{n} v_i}$;

$\mathbf{x}=(x_1,\ldots,x_n)'$ — the (unknown) apportionment vector, where $x_i\geq 0$, $\sum_{i=1}^{n} x_i=S$, are the parties' integer-valued numbers of parliamentary seats.

We measure the apportionment accuracy in absolute and relative terms. The absolute accuracy is the deviation of the parties' shares of parliamentary seats from the quotas, and the relative accuracy is the maximum of these deviations relative to the quotas:

$$\text{Absolute apportionment accuracy} \quad \varepsilon \quad = \quad \max_i \underbrace{\left|\frac{x_i}{S}-q_i\right|}_{\substack{i\text{th faction}\\ \text{fitting error}}} \tag{2}$$

[4]In this paper, all vectors are column-vectors.

$$\text{Relative apportionment accuracy} \quad \varepsilon \;=\; \max_i \underbrace{\frac{\left|\frac{x_i}{S} - q_i\right|}{q_i}}_{\substack{i\text{th faction}\\ \text{fitting error}\\ \text{relative to}\\ \text{the quota size}}} = \max_i \left|\frac{x_i}{q_i S} - 1\right| . \tag{3}$$

For both accuracy measures, the apportionment optimization problem is formulated in two ways described in the following subsections.

3.1 Minimizing parliament size S for a given apportionment accuracy ε

One way of optimizing apportionment is as follows: Given an apportionment accuracy ε, find the apportionment $\mathbf{x}$ while minimizing the parliament size S. In mix-member proportional representation systems this task is often considered subject to constraints like minimum party seats, e.g. reserved for local representatives elected in constituencies.

The apportionment accuracy constraint in absolute terms (2) means that the ith party's share of parliamentary seats $x_i / \sum_{j=1}^n x_j$ is within the party's parliament quota $q_i \pm \varepsilon$:

$$-\varepsilon \le \frac{x_i}{\sum_{j=1}^n x_j} - q_i \le \varepsilon, \quad i = 1, \ldots, n. \tag{4}$$

The right inequality in (4) can be rewritten as

$$x_i - (q_i + \varepsilon) \sum_{j=1}^n x_j \le 0, \quad i = 1, \ldots, n.$$

In the matrix form, these n inequalities look as follows:

$$\begin{pmatrix} (1-(q_1+\varepsilon))x_1 & -(q_1+\varepsilon)x_2 & \cdots & -(q_1+\varepsilon)x_n \\ -(q_2+\varepsilon)x_1 & (1-(q_2+\varepsilon))x_2 & \cdots & -(q_2+\varepsilon)x_n \\ \cdots & \cdots & \cdots & \cdots \\ -(q_n+\varepsilon)x_1 & -(q_n+\varepsilon)x_2 & \cdots & (1-(q_n+\varepsilon))x_n \end{pmatrix} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$\underbrace{\begin{pmatrix} 1-(q_1+\varepsilon) & -(q_1+\varepsilon) & \cdots & -(q_1+\varepsilon) \\ -(q_2+\varepsilon) & 1-(q_2+\varepsilon) & \cdots & -(q_2+\varepsilon) \\ \cdots & \cdots & \cdots & \cdots \\ -(q_n+\varepsilon) & -(q_n+\varepsilon) & \cdots & 1-(q_n+\varepsilon) \end{pmatrix}}_{\mathbf{A}_1} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Similarly, the left inequalities in (4) are represented in the matrix form as well:

$$\underbrace{\begin{pmatrix} -1+(q_1-\varepsilon) & q_1-\varepsilon & \cdots & q_1-\varepsilon \\ q_2-\varepsilon & -1+(q_2-\varepsilon) & \cdots & q_2-\varepsilon \\ \cdots & \cdots & \cdots & \cdots \\ q_n-\varepsilon & q_n-\varepsilon & \cdots & -1+(q_n-\varepsilon) \end{pmatrix}}_{\mathbf{A}_2} \cdot \mathbf{x} \le \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thereby, we reduce our task to the mixed-integer linear programming problem solvable by function `intlinprog` from the MATLAB Optimization Toolbox:[5]

$$\min_{\mathbf{x}} \underbrace{\mathbf{1}'\cdot\mathbf{x}}_{\substack{\text{size of the}\\ \text{parliament}}} \quad \text{subject to} \quad \begin{cases} \mathbf{x} \ \text{is integer-valued} & \\ \begin{pmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{pmatrix} \cdot \mathbf{x} \leq \mathbf{0} & \text{(Constraint: absolute apportionment accuracy)} \\ \mathbf{x} \geq \mathbf{m} & \text{(Constraint: minimum number of party seats)} \end{cases} \tag{5}$$

where

$\mathbf{1}$ is the n-vector of 1s,

$\mathbf{x}$ is the apportionment n-vector of seats allocated to n parties,

$\mathbf{m}$ is the non-negative integer-valued n-vector of parties' minimum seats, e.g. $\mathbf{m} = (100, 80, \ldots, 0)'$,

$\mathbf{0}$ is the $2n$-vector of 0s.

The apportionment accuracy constraint in relative terms (3) implies the inequalities

$$-\varepsilon \leq \frac{\frac{x_i}{\sum_{j=1}^n x_j} - q_i}{q_i} \leq \varepsilon, \quad i = 1, \ldots, n. \tag{6}$$

The right inequalities in (6) can be rewritten as

$$x_i - (q_i + \varepsilon q_i) \sum_{j=1}^n x_j \leq 0, \quad i = 1, \ldots, n,$$

which by analogy with the previous case is equivalent to

$$\underbrace{\begin{pmatrix} 1-(q_1+\varepsilon q_1) & -(q_1+\varepsilon q_1) & \cdots & -(q_1+\varepsilon q_1) \\ -(q_2+\varepsilon q_2) & 1-(q_2+\varepsilon q_2) & \cdots & -(q_2+\varepsilon q_2) \\ \cdots & \cdots & \cdots & \cdots \\ -(q_n+\varepsilon q_n) & -(q_n+\varepsilon q_n) & \cdots & 1-(q_n+\varepsilon q_n) \end{pmatrix}}_{\mathbf{R}_1} \cdot \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\mathbf{x}} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The left inequalities in (6) are represented in the matrix form as well:

$$\underbrace{\begin{pmatrix} -1+(q_1-\varepsilon q_1) & q_1-\varepsilon q_1 & \cdots & q_1-\varepsilon q_1 \\ q_2-\varepsilon q_2 & -1+(q_2-\varepsilon q_2) & \cdots & q_2-\varepsilon q_2 \\ \cdots & \cdots & \cdots & \cdots \\ q_n-\varepsilon q_n & q_n-\varepsilon q_n & \cdots & -1+(q_n-\varepsilon q_n) \end{pmatrix}}_{\mathbf{R}_2} \cdot \mathbf{x} \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

[5] Matrices $\mathbf{A}_1, \mathbf{A}_2$ can be generated using the MATLAB function `diag(x)` to create a diagonal matrix with vector $\mathbf{x}$ at its diagonal and the operation of adding a scalar to all elements of a vector or matrix:

$$\begin{aligned} \mathbf{A}_1 &= \mathtt{diag}(\mathbf{1}) - \mathtt{diag}(\mathbf{q}+\varepsilon)\cdot\mathbf{1} \\ \mathbf{A}_2 &= -\mathtt{diag}(\mathbf{1}) + \mathtt{diag}(\mathbf{q}-\varepsilon)\cdot\mathbf{1}, \end{aligned}$$

where $\mathbf{1}$ and $\mathbf{1}$ are the n-vector and $n \times n$-matrix of 1s, respectively.

The task is thereby reduced to the mixed-integer linear programming problem solvable by function `intlinprog` from the MATLAB Optimization Toolbox:[6]

$$\min_{\mathbf{x}} \underbrace{\mathbf{1}' \cdot \mathbf{x}}_{\substack{\text{size of the}\\ \text{parliament}}} \quad \text{subject to} \begin{cases} \mathbf{x} \ \text{ is integer-valued} & \\ \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \end{pmatrix} \cdot \mathbf{x} \leq \mathbf{0} & \text{(Constraint: relative apportionment accuracy)} \\ \mathbf{x} \geq \mathbf{m} & \text{(Constraint: minimum number of party seats)} \end{cases}$$

3.2 Directed search for the apportionment of S seats with the best absolute accuracy

The second way to optimize apportionment is as follows: Given a parliament size S, find the apportionment $\mathbf{x}$ while minimizing the accuracy threshold ε. If the accuracy is measured in absolute terms (2) then we have the problem

$$\underbrace{\min_{\mathbf{x}} \max_{i} \underbrace{\left| \frac{x_i}{S} - q_i \right|}_{\substack{i\text{th faction}\\ \text{absolute}\\ \text{fitting error}}}}_{\substack{\text{absolute}\\ \text{apportionment}\\ \text{accuracy}}} \quad \text{subject to} \begin{cases} \mathbf{x} \ \text{ is integer-valued} \\ \mathbf{x} \geq \mathbf{m} \quad \text{(Constraint: minimum number of party seats)} \end{cases} \tag{7}$$

Since the number of seats S is fixed, the only way to reduce the absolute apportionment accuracy threshold in (7) is to move seats from one party faction to another. Specifically, we find the party faction j with the greatest deviation from its quota, i.e.

$$j : \left| \frac{x_j}{S} - q_j \right| = \max_i \left| \frac{x_i}{S} - q_i \right| \, .$$

If the jth party quota is exceeded, i.e. $\frac{x_j}{S} > q_j$, then we move one its seat to another party faction, trying to maximally reduce $\max_i \left| \frac{x_i}{S} - q_i \right|$; we have to try all party factions because for small factions adding even one seat can significantly change the deviation from the quota. If the party quota is not reached, i.e. $\frac{x_j}{S} < q_j$, then we increase x_j by moving one seat from some other faction, trying to maximally reduce $\max_i \left| \frac{x_i}{S} - q_i \right|$; we also have to try all party factions. If no such seat exchange improves the absolute apportionment accuracy then the current apportionment is indeed optimal. Otherwise, we repeat the cycle as long as the absolute apportionment accuracy can be improved.

[6] Matrices $\mathbf{R}_1, \mathbf{R}_2$ can be generated in the same way as described in the previous footnote:

$$\begin{aligned} \mathbf{R}_1 &= \texttt{diag}(\mathbf{1}) - \texttt{diag}(\mathbf{q} + \varepsilon\mathbf{q}) \cdot \mathbf{1} \\ \mathbf{R}_2 &= -\texttt{diag}(\mathbf{1}) + \texttt{diag}(\mathbf{q} - \varepsilon\mathbf{q}) \cdot \mathbf{1} \, . \end{aligned}$$

3.3 Maximizing the relative apportionment accuracy using adjustment vote weights

To find the apportionment of S parliamentary seats with the maximum relative accuracy, we have to solve the optimization problem (7), replacing $\dfrac{x_i}{S} - q_i$ by $\dfrac{\frac{x_i}{S} - q_i}{q_i}$:

$$\underbrace{\min_{\mathbf{x}} \max_i \underbrace{\left|\frac{x_i}{q_i S} - 1\right|}_{\substack{i\text{th faction}\\ \text{relative}\\ \text{fitting error}}}}_{\substack{\text{relative}\\ \text{apportionment}\\ \text{accuracy}}} \quad \text{subject to} \begin{cases} \mathbf{x} \text{ is integer-valued} \\ \mathbf{x} \geq \mathbf{m} \quad \text{(Constraint: minimum number of party seats)} \end{cases} \tag{8}$$

Finding the solution to (8) is however less straightforward because one seat has different significance for large and small factions. To overcome this complication, we consider *adjustment vote weights*, with which the voting power of each party faction is made equal to the party's parliamentary quota. For instance, if a parliament has 99 seats and two party factions have equal quotas $\mathbf{q} = (0.5, 0.5)'$, the inevitable apportionment is $\mathbf{x} = (49, 50)'$, i.e. one party is under- and another over-represented. To ensure parity, the vote weight of the members of the underrepresented faction is made 50/49, which thereby equalizes the factions' voting power. Thus, if q_i is the quota of the ith party then the adjustment vote weight w_i has to satisfy the equation

$$w_i x_i = S q_i \quad \Leftrightarrow \quad w_i = S q_i / x_i \ , \quad i = 1, \ldots, n \ . \tag{9}$$

The vector form of these equations is as follows:[7]

$$\mathbf{w} = S\mathbf{q}./\mathbf{x} \quad \text{(the adjustment vote weight vector).}$$

In our model, the members of the most overrepresented faction (with the smallest adjustment vote weight) are assigned the vote weight = 1, and the adjustment vote weights for the members of other factions are obviously > 1, i.e. we redefine[8]

$$\mathbf{w} = \frac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]} \quad \text{(the normalized adjustment vote weight vector).} \tag{10}$$

The following simple proposition enables improving the relative apportionment accuracy by equalizing adjustment vote weights.

Theorem 1 (Maximizing relative apportionment accuracy by equalizing adjustment vote weights) *While improving the relative apportionment accuracy (= minimizing ε), the range of adjustment vote weights is minimized, and vice versa.*

The directed search for the apportionment with the best relative accuracy is similar to the one described in Section 3.2 with the only difference that the largest relative deviation from the quota is recognized by the greatest adjustment vote weight — for the most underrepresented party, and by the vote weight 1 — for the most overrepresented party.

Thus, the optimization problem of minimizing *relative* deviations of factions from quotas has a dual formulation in terms of minimizing the range of adjustment vote weights. Since a greater range of adjustment vote weights means a higher inequality of members of parliament, the problem of minimizing *relative* deviations from quotas can be interpreted as *equalizing individual powers* of the members of parliament.

---

[7]The following operation ./ is the element-by-element division of vectors, e.g. $(25, 16)./(5, 2) = (5, 8)$. The operation .* (used below) is the element-by-element multiplication of vectors, e.g. (25,16).*(5,2)=(125,32).

[8]The notation like min[**x**] means the corresponding operation on the elements of the column vector in the brackets.

3.4 Lexicographic optimization

The algorithm of directed search can be used under various constraints and even with an option of starting from an arbitrary apportionment. This property is important in the *double lexicographic optimization* when two optimization criteria are successively applied. For example, there may be several optimal apportionments with respect to the absolute accuracy criterion, and then we apply the relative optimization to select among them the best one with regard to the relative accuracy criterion. Conversely, after we have found an optimal apportionment for the relative accuracy criterion, we try to improve it further with respect to the absolute accuracy criterion.

Another advantage of the algorithm of directed search is the possibility to add or remove parliamentary seats one-by-one, i.e. gradually increase/decrease the size of the parliament, taking care of optimality of the resulting apportionment. Such a discrete optimization algorithm turns out to be flexible, computationally efficient and well suited to analyze the D'Hondt and Sainte-Laguë methods of apportionment.

3.5 Computer implementation

We will compare four apportionment methods: that of D'Hondt (DH), Sainte-Laguë (SL), optimization with the Absolute Accuracy criterion (AA), and optimization with the Relative Accuracy criterion (RA). For this purpose, we define the vector-function f, which to arguments $\mathbf{q}$, $\mathbf{m}$, 'Method' and S puts into correspondence the apportionment vector

$$\mathbf{x} = f(\mathbf{q}, \mathbf{m}, \text{Method}, S)\ , \qquad \text{Method = DH, SL, AA, RA,} \qquad S = S_{\min}\ ,\ S_{\min}+1,\ \ldots\ ,\ S_{\max}\ .$$

For Methods DH and SL, f is computed by algorithm (1), and for Methods AA and RA, we use the directed search algorithm to solve (7) and (8), respectively. Both methods AA and RA apply the two-level lexicographic optimization as described in Section 3.4; for example, if the absolute accuracy criterion results in several optimal solutions, the best one among them is selected with respect to the relative accuracy criterion.

The apportionment $\mathbf{x}$ is used to derive further n-vectors, which constitute the tables below:

$100\,\mathbf{q}$ — vector of parliamentary party quotas in %,

$\mathbf{x}-\mathbf{m}$ — vector of adjustment (leveling) seats,

$100\dfrac{\mathbf{x}}{S}$ — vector of the faction sizes, in % of Bundestag seats,

$\mathbf{x}-S\,\mathbf{q}$ — vector of factions' absolute deviations from the quotas, in number of seats; cf. with (7),

$100\left(\dfrac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$ — vector of factions' relative deviations from the quotas, in %; cf. with (8),

$\mathbf{w} = \dfrac{S\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]}$ — vector of faction members' adjustment vote weights normalized, that is, with the minimum adjustment vote weight = 1,

$\mathbf{x}.*\mathbf{w}$ — vector of faction's total number of adjusted votes,

$100\dfrac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$ — vector of faction's vote power in %.

The computer program is written in MATLAB. It outputs LaTeX-files with tables and eps-files with figures displayed below.

4 Example: Apportionment of the 2021 German Bundestag

4.1 Public concerns about the apportionment of the Bundestag

To illustrate our approach, we apportion the German Bundestag basing on the 2021 federal elections. Recall that German citizens cast two votes. In the first ballot, a specific person, a *direct mandate* holder, is elected by a simple majority in each of 299 constituencies. Thereby, half of the 598 nominal Bundestag seats are filled.

The second vote is cast for one of the registered parties. It has three functions. The first function is rejecting the parties that receive less than 5% of the nationwide votes. It does not apply if a party gets at least three direct mandates or it represents a recognized ethnical minority — currently Danes, Frisians, Sorbs, and Romani people.

Next, the second vote is used within each of the 16 German federal states, determining the party quotas for the *state* representation in the Bundestag (taking into account that the sizes of state representations must be proportionate to the number of states' citizens). However, the state's direct mandate holders from a certain party can be too numerous, resulting in *overhang mandates*, which are compensated by giving additional leveling (adjustment) mandates to other parties, increasing the total number of the state's deputies. If the direct mandates are too few to reach the state's party quota, some 'vacancies' (not all — to facilitate adjustments at the federal level) are filled from the *party list* defined by the party.

The third function of the second vote is determining the final ratio of party factions in the Bundestag. For this purpose, the remaining 299 nominal Bundestag seats are used. If these 299 seats are insufficient to achieve the Bundestag factions' proportionality to the party votes within the required accuracy of 0.5 seat (which enables rounding to integers), leveling seats are added; for more details see [Bundeswahlleiter 2021] and [Bundestag 2023, Section 'Election'].

To avoid numerous leveling seats, three unadjusted seats are currently tolerated, but even this exception does not constrain the Bundestag's uncontrolled growth. Historically, the 2005, 2009, 2013, 2017 and 2021 German Bundestags with 598 nominal seats required additionally 16, 24, 33, 111 and 138 adjustment seats, respectively. Thereby, the actual 2021 Bundestag with three unadjusted mandates has as many as 736 members (if correctly adjusted it would have 795 members), which makes it the second largest parliament worldwide after China. Such a growth makes the Bundestag more expensive for taxpayers: its annual budget is already approaching billion Euro [Finthammer 2018]. In 2016, Norbert Lammert, then the 12th president of the Bundestag, proposed to restrict it to 630 members by allocating mandates strictly according to state quotas, which should be proportional to their population [Roßner 2016]. This idea found no approval among the German parties, neither large nor small [Finthammer 2018]. Only in October 2019, after predictions that the next Bundestag could exceed 800 seats, did some 100 German experts in constitutional law write an open letter suggesting to constrain its size by reducing the number of effective constituencies, and the Bundestag vice-president, Thomas Oppermann, called for such a reform without delay [Spiegel online 2019, Zeit online 2019].

These and other proposals require a profound change in the existing election system. But a general mathematical solution to the problem does not require such changes and is much simpler. The unfettered growth of the German Bundestag — caused by allotting too many direct mandates to parties that received too few second votes — can be prevented by introducing *adjustment vote weights* for Bundestag members. The adjustment vote weights could make numerous adjustment seats unnecessary and the 598 nominal Bundestag seats sufficient under most circumstances. This issue, among other things, is considered below.

4.2 Official processing of the 2021 Bundestag election results

Table 1 characterizes the eight parties that are entitled to the 2021 Bundestag seats. The computation of their minimum entitlement is beyond the scope of this paper. We therefore take it for granted, focusing on finding the appropriate size of the Bundestag and the final allocation of its seats to the parties.

Table 1: Eight German parties entitled to the 2021 Bundestag seats

	Party logo	Party description	Valid votes		Quota in the Bundestag	Minimum seat entitlement
			Number	%	%	
1		SPD, Sozial-demokratische Partei Deutschlands (Social Democratic Party), founded in 1863.	11,955,434	25.79	28.173	170
2		CDU, Germany's main conservative party, Christlich Demokratische Union Deutschlands (Christian Democratic Union of Germany), founded in 1950, makes a political union with the CSU.	8,775,471	18.93	20.679	122
3		GRÜNE, BÜNDNIS 90/DIE GRÜNEN (Alliance 90/The Greens), founded in 1993 as the merger of DIE GRÜNEN (West Germany) and BÜNDNIS 90 (East Germany), both with a social-democratic background.	6,852,206	14.78	16.147	94
4		FDP, Freie Demokratische Partei (Free Democratic Party) is founded in 1948, stands for political liberalism and wants to strengthen the freedom, self-determination and responsibility of the individual within the framework of the social market economy. It has been represented in the Bundestag since 1949 — with an interruption from 2013 to 2017.	5,319,952	11.47	12.536	76
5		AfD, Alternative für Deutschland (Alternative for Germany) is founded in 2013 to protest against financial aid for economically struggling EU member states. By criticizing the asylum and refugee policy, it has increasingly distinguished itself as a right-wing populist protest party. In 2021, it was represented in all German state parliaments and in the Bundestag.	4,803,902	10.36	11.320	69
6		CSU, Bavarian conservative party, Christlich-Soziale Union in Bayern (Christian Social Union of Bavaria), founded in 1945, makes a political union with the CDU.	2,402,827	5.18	5.662	45
7		DIE LINKE (The Left), founded in 2007 as the merger of East German communists and the Electoral Alternative for Labor and Social Justice (WASG), a left-wing breakaway from the SPD. It advocates democratic socialism and the expansion of the welfare state and sees itself as a peace party that advocates nonviolence. It or its predecessor PDS has been represented in the Bundestag since 1990.	2,270,906	4.90	5.351	32
8		SSW, Südschleswigscher Wählerverband (South Schleswig Association of Voters) is founded in 1948. It represents the political interests of the Danish minority and the Frisian ethnic group and is therefore exempt from the threshold clause. It has been continuously represented in the Schleswig-Holstein state parliament since 1958. It is running for the first time since 1961 in the federal election.	55,578	0.12	0.131	1
	Other parties	....	3,925,737	8.47	0	0
	Total		46,362,013	100.00	100.000	609

The officially adopted Schepers' version of the Satinte-Laguë method in the form of *divisor procedure* is traced in Tables 2 and 3, which are screenshots of Tables 6.1.4 and 6.2.3 from the official electoral report [Bundeswahlleiter 2021, pp. 421 and 451] with the English translation of their headings added above.[9] In Table 2, Column 1 displays the minimum number of Bundestag seats, to which a party is entitled according to the German electoral rules. The bottom row, 'Total', summarizes the columns; thus, the 2021 Bundestag must have at least 609 seats. Column 2 warns of threatening overhang mandates, which emerge at the level of German states but are mostly compensated at the federal level. Column 3 contains the number of valid second votes cast for the party. Column 4, 'Divisor', indicates the number of votes needed by a party to get one Bundestag seat. The divisor must adjust the rounded quotient of the party votes divided by the divisor to the party's Bundestag quota with the accuracy of ±0.5 seat (Columns 5 and 6 give the exact and rounded quotients, respectively). For example,

$$\begin{aligned}\text{Minimum number of seats for the FDP} &= \frac{5\,319\,952}{57\,898}\\ &= 91.884\\ &\approx 92 \text{ seats}.\end{aligned}$$

Moreover, to reduce the total number of Bundestag seats, three overhang mandates are tolerated and for this reason are excluded from consideration (in Column 6, CSU has 42 seats instead of 45 in Column 1). Column 8 displays the final allocation of Bundestag seats to the parties, and Columns 9–10 show how this apportionment relates to the 598 nominal Bundestag seats.

The divisor in Column 4 is found by the Sainte-Laguë method in the form of divisor procedure. Its application to the outcomes of the 2021 elections is traced in Table 3. Column 1 displays the number of votes cast for the parties. Because of final rounding, the party's minimum number of seats for intermediate computations is reduced by 0.5, as in Column 2. Column 3 reveals that the 11 'threatening overhangs' of CSU are in fact real because the CSU with no overhangs should have 33.5 seats instead of 44.5. The divisor procedure begins in Columns 4–5 where only the parties with threatening overhangs are retained; cf. with Column 2 of Table 2. In Columns 6, 8 and 9, the party factions with still threatening overhangs are successively reduced by one seat (since AfD has one threatening overhang, its faction is reduced only once and then it is no longer considered), and their individual divisors are recalculated in Columns 7, 9 and 11. Since only three overhangs are tolerated, the procedure stops at Columns 10–11 and the smallest divisor, in this case for the CSU, is selected, rounded and used in Column 4 of Table 2.

Our implementation of the Saint-Laguë method is more straightforward. The first task is finding the appropriate Bundestag size, which we explain referring to Table 4. Columns 1–2 show the conversion of the number of votes cast for the party into the party quota in the Bundestag. For example,

$$\begin{aligned}q_6 = \text{CSU quota} &= \frac{\text{Number of votes for the CSU}}{\text{Total number of votes for the parties eligible for Bundestag seats}}\\ &= \frac{2\,402\,827}{42\,436\,276}\\ &\approx 0.0566 \quad (= 5.66\%).\end{aligned}$$

Column 3 (Section 'No overhangs') shows the virtual minimum number of party seats that after rounding guarantees the actual minimum number of seats — for this purpose the latter is reduced by 0.5. Therefore, Column 3 is equal to Column 1 of Table 2 reordered and reduced by 0.5. To provide the required absolute apportionment accuracy of 0.5 seat, the Bundestag size S must be sufficiently large to allocate the minimum party seats $m_i - 0.5$ within the party quotas q_iS, i.e.

$$m_i - 0.5 \le q_iS \quad \Leftrightarrow \quad S \ge \frac{m_i - 0.5}{q_i}, \quad i = 1, \dots, n\,.$$

[9] Five AfD members and one SSW member belong to no Bundestag faction, implying the difference between the election results and the composition of the Bundestag shown in [Deutscher Bundestag 2023].

Table 2: Increasing the total number of seats for the parties (screenshot of a table from the official German election report with an English translation of its heading added above)

Party	Minimum seat entitlement (Maximum of sum of quota of seats and sum of minimum number of seats	Threatening overhang	Second votes	Divisor	Seats after increase: unrounded	Seats after increase: rounded	Remaining overhang (Column 7)	Total seats (Column 8)	Seats according to quotas of seats (Column 9)	Increase by … seats (Difference between Columns 8 and 9)
	1	2	3	4	5	6	7	8	9	10
Partei	**Mindestsitzanspruch** (Maximum aus Summe Sitzkontingente und Summe Mindestsitzzahlen)	**Drohender Überhang**	**Zweitstimmen**	**Divisor**	**Sitze nach Erhöhung: ungerundet**	**Sitze nach Erhöhung: gerundet**	**Verbleibender Überhang** (Spalte 7)	**Sitze insgesamt** (Spalte 8)	**Sitze nach Sitzkontingenten** (Spalte 9)	**Erhöhung um ... Sitze** (Differenz aus Spalten 8 und 9)
CDU	122	12	8.775.471		151,567	**152**	–	152	122	30
SPD	170	10	11.955.434		206,491	**206**	–	206	170	36
AfD	69	1	4.803.902		82,971	**83**	–	83	69	14
FDP	76	–	5.319.952	57.898	91,884	**92**	–	92	76	16
DIE LINKE	32	–	2.270.906		39,222	**39**	–	39	32	7
GRÜNE	94	–	6.852.206		118,349	**118**	–	118	94	24
CSU	45	11	2.402.827		41,501	**42**	**3**	45	34	11
SSW	1	–	55.578		0,959	**1**	–	1	1	–
Insgesamt	**609**	**34**	**42.436.276**			**733**	**3**	**736**	**598**	**138**

Source: [Bundeswahlleiter 2021, Section “6.1.4 Oberverteilung: Erhöhung der Gesamtzahl der Sitze für die Parteien”, p. 421]

Table 3: Finding the divisor range and the final divisors (screenshot of a table from the official German election report with an English translation of its heading added above)

Party	Second votes	Minimum entitlement of parties to seats: without overhangs: −0.5	Party divisor	with overhangs: −0.5	Party divisor	−1.5	Party divisor	−2.5	Party divisor	−3.5	Party divisor
	1	2	3	4	5	6	7	8	9	10	11
Partei	**Zweitstimmen**	**Mindestsitzanspruch der Parteien: ohne Überhang: -0,5**	**Parteien-Divisor**	**mit Überhang: -0,5**	**Parteien-Divisor**	**-1,5**	**Parteien-Divisor**	**-2,5**	**Parteien-Divisor**	**-3,5**	**Parteien-Divisor**
CDU	8.775.471	121,5	72.226,099	121,5	72.226,099	120,5	72.825,485	119,5	73.434,904	118,5	74.054,608
SPD	11.955.434	169,5	70.533,534	169,5	70.533,534	168,5	70.952,131	167,5	71.375,725	166,5	71.804,408
AfD	4.803.902	68,5	70.129,956	68,5	70.129,956	67,5	71.168,919		–		–
FDP	5.319.952	75,5	70.462,940		–		–		–		–
DIE LINKE	2.270.906	31,5	72.092,254		–		–		–		–
GRÜNE	6.852.206	93,5	73.285,626		–		–		–		–
CSU	2.402.827	33,5	71.726,179	44,5	53.996,112	43,5	55.237,402	42,5	56.537,106	41,5	57.899,446
SSW	55.578	0,5	111.156		–		–		–		–
Insgesamt	**42.436.276**										

Source: [Bundeswahlleiter 2021, Section “6.2.3 Ermittlung der Divisorspanne und des endgültigen Divisors”, p. 451]

Table 4: Minimum Bundestag with successively eliminated overhangs

	Elections result		No overhangs			-1 overhang			-2 overhangs			-3 overhangs		
	Votes received	Quota of the party in the Bundestag, in %	Minimum seat entitlement	Bundestag size with minimum party seats within quota	Minimum integer Bundestag size	Minimum seat entitlement	Bundestag size with minimum party seats within quota	Minimum integer Bundestag size	Minimum seat entitlement	Bundestag size with minimum party seats within quota	Minimum integer Bundestag size	Minimum seat entitlement	Bundestag size with minimum party seats within quota	Minimum integer Bundestag size
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
SPD	11955434 →	28.17	169.5	601.6	602	169.5	601.6	602	169.5	601.6	602	169.5	601.6	602
CDU	8775471 →	20.68	121.5	587.5	588	121.5	587.5	588	121.5	587.5	588	121.5	587.5	588
GRÜNE	6852206 →	16.15	93.5	579.1	580	93.5	579.1	580	93.5	579.1	580	93.5	579.1	580
FDP	5319952 →	12.54	75.5	602.2	603	75.5	602.2	603	75.5	602.2	603	75.5	602.2	603
AfD	4803902 →	11.32	68.5	605.1	606	68.5	605.1	606	68.5	605.1	606	68.5	605.1	606
CSU	2402827 →	5.66	44.5	$\boxed{785.9}$	$\boxed{786}$	43.5	$\boxed{768.3}$	$\boxed{769}$	42.5	$\boxed{750.6}$	$\boxed{751}$	41.5	$\boxed{732.9}$	$\boxed{733}$
LINKE	2270906 →	5.35	31.5	588.6	589	31.5	588.6	589	31.5	588.6	589	31.5	588.6	589
SSW	55578 →	0.13	0.5	381.8	382	0.5	381.8	382	0.5	381.8	382	0.5	381.8	382
Sum/ $\boxed{\text{Max}}$	42436276 →	100.00	605.0	$\boxed{785.9}$	$\boxed{786}$	604.0	$\boxed{768.3}$	$\boxed{769}$	603.0	$\boxed{750.6}$	$\boxed{751}$	602.0	$\boxed{732.9}$	$\boxed{733}$

Representing these inequalities for eight parties in the vector form, we have

$$\begin{aligned}
S &\geq \max[(\mathbf{m}-0.5)./\mathbf{q}] \\
&= \max\left[\begin{pmatrix} 170-0.5 \\ 122-0.5 \\ 94-0.5 \\ 76-0.5 \\ 69-0.5 \\ 45-0.5 \\ 32-0.5 \\ 1-0.5 \end{pmatrix} ./ \begin{pmatrix} 0.28173 \\ 0.20679 \\ 0.16147 \\ 0.12536 \\ 0.11320 \\ 0.05662 \\ 0.05351 \\ 0.00131 \end{pmatrix}\right] \\
&= \max \underbrace{\left[\begin{pmatrix} 601.6399 \\ 587.5526 \\ 579.0549 \\ 602.2655 \\ 605.1237 \\ 785.9414 \\ 588.6750 \\ 381.6794 \end{pmatrix}\right]}_{\min S \text{ for each party}} \\
&\approx 786\,.
\end{aligned} \tag{11}$$

This is reflected in Columns 4–5, which show, respectively, the fraction-valued and the upward rounded Bundestag size that enables the given party quota accommodate all the minimum seats the party is entitled to. The largest Bundestag size is in the 6th row for the CSU (boxed), indicating that the CSU faction is most overrepresented, so the first tolerated must be the overhang of the CSU. Having removed one CSU's overhang from consideration, the same analysis is repeated in Section '-1 overhang' where the number of CSU minimum seats is reduced to 43.5 (Column 6). The most overrepresented is again the CSU, and the next tolerated overhang must be that of the CSU. (If the faction of some other party implied the largest Bundestag, we would reduce it instead of the CSU faction.) Proceeding in this way,

Table 5: Apportionment of the 2021 Bundestag with 733 seats and no overhangs

	Official apportionment by the **Sainte-Laguë** method									
	Party quota in Bundestag, in % $100\mathbf{q}$	Minimum seat entitlement $\mathbf{m}$	Adjustment seats $\mathbf{x}-\mathbf{m}$	Seats (apportionment) $\mathbf{x}$	Seats, in % of the Bundestag seats $100\frac{\mathbf{x}}{S}$	Absolute deviation from the quota, in number of seats $\mathbf{x}-S\mathbf{q}$	Relative deviation from the quota, in % of the quota $100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	Adjustment vote weight $\mathbf{w}$	Party's votes (Seats × Vote weight) $\mathbf{x}.*\mathbf{w}$	Party's share of power, in % $100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	1	2	3	4	5	6	7	8	9	10
SPD	28.173	170 +	36	= 206	→ 28.104	$\boxed{-0.506}$	−0.245	1.044	$\boxed{215.111}$	→ 28.173
CDU	20.679	122 +	30	= 152	→ 20.737	0.422	0.278	1.039	157.895	→ 20.679
GRÜNE	16.147	94 +	24	= 118	→ 16.098	−0.358	−0.302	1.045	123.290	→ 16.147
FDP	12.536	76 +	16	= 92	→ 12.551	0.109	0.118	1.040	95.720	→ 12.536
AfD	11.320	69 +	14	= 83	→ 11.323	0.022	0.027	1.041	86.435	→ 11.320
CSU	5.662	42 +	0	= 42	→ 5.730	$\boxed{0.496}$	1.195	1.029	43.233	→ 5.662
LINKE	5.351	32 +	7	= 39	→ 5.321	−0.225	$\boxed{-0.574}$	$\boxed{1.048}$	40.860	→ 5.351
SSW	0.131	1 +	0	= 1	→ 0.136	0.040	$\boxed{4.167}$	$\boxed{1.000}$	$\boxed{1.000}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	606 +	127	= 733	→ 100.000	$\boxed{1.002}$	$\boxed{4.741}$	$\boxed{0.048}$	$\boxed{214.111}$	→ 100.000

we find that all the three tolerated overhangs are that of the CSU, implying the Bundestag size

$$S \geq (m_6 - 3 - 0.5) \cdot q_6 = (42 - 0.5)/0.05662 = 732.9566 \approx 733 \, .$$

Now we allocate 733 seats by the Sainte-Laguë method (1). The resulting apportionment in Column 4 of Table 5 is equal to the apportionment in Column 6 of Table 2. The accuracy of fitting the party factions to party quotas in Column 6 of Table 5 is within the required accuracy of 0.5 seats (a negligible inaccuracy of −0.506 is inherent in the SPD faction). After the 733 Bundestag seats without three CSU overhangs have been allocated, the final apportionment is obtained by adding three 'tolerated' overhangs to the CSU faction.

Alternatively, the Sainte-Laguë algorithm for minimum reserved seats (described in the last paragraph of Section 2) directly allocates 736 seats in one run. The resulting apportionment in Column 4 of the upper block of Table 6 is the same as in Column 8 of Table 2 and in Column 4 of Table 5 + three seats for the CSU. Due to three unadjusted mandates, the apportionment accuracy drops sharply (cf. Columns 6 in Tables 5 and 6). Now the CSU faction exceeds its quota by 3.326 seats, and the SPD is underrepresented by 1.351 seat — instead of the tolerated inaccuracy of 0.5 seat.

4.3 Non-optimality of the Sainte-Laguë method of apportionment

Now we compare the four apportionment methods for two Bundestag sizes: 736 (the actual size) and 630 (as suggested by Norbert Lammert).[10] The computations for each Bundestag size are displayed in Tables 6 and 7, each consisting of four similar blocks for four apportionment methods (with columns numbered end-to-end). The maximum deviations from the quotas, both positive and negative, are boxed, and their max-min ranges are in the bottom row and boxed as well.

Let us first consider the official Sainte-Laguë apportionment of the Bundestag with 736 seats (including three unadjusted CSU overhangs) displayed in the upper block of Table 6. Column 1 shows the parties' Bundestag quotas in % of seats, that is, $100\mathbf{q}$. The vector $\mathbf{m}$ of minimum entitlement to seats, which we take for granted, is displayed in Column 2. Column 3 shows the vector $\mathbf{x}-\mathbf{m}$ of adjusted seats, Column 4 — the computed apportionment $\mathbf{x}$, and Column 5 — the apportionment in % of the Bundestag seats $100\frac{\mathbf{x}}{S}$. Columns 6 shows the absolute deviations of party factions from their quotas in the number of seats $\mathbf{x}-S\mathbf{q}$. The boxes highlight the minimum and maximum deviations from

[10]For the Bundestag sizes of 609 and 598 seats see tables in [Tangian 2022].

Table 6: Apportionment of the 2021 Bundestag with 736 seats including three unadjusted seats

		Official apportionment by the **Sainte-Laguë** method								
	Party quota in Bundestag, in % $100\mathbf{q}$	Minimum seat entitlement $\mathbf{m}$	Adjustment seats $\mathbf{x}-\mathbf{m}$	Seats (apportionment) $\mathbf{x}$	Seats, in % of the Bundestag seats $100\frac{\mathbf{x}}{S}$	Absolute deviation from the quota, in number of seats $\mathbf{x}-S\mathbf{q}$	Relative deviation from the quota, in % of the quota $100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	Adjustment vote weight $\mathbf{w}$	Party's votes (Seats × Vote weight) $\mathbf{x}.*\mathbf{w}$	Party's share of power, in % $100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	1	2	3	4	5	6	7	8	9	10
SPD	28.173	170 +	36	= 206	→ 27.989	$\boxed{-1.351}$	−0.651	1.087	$\boxed{223.901}$	→ 28.173
CDU	20.679	122 +	30	= 152	→ 20.652	−0.199	−0.131	1.081	164.346	→ 20.679
GRÜNE	16.147	94 +	24	= 118	→ 16.033	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76 +	16	= 92	→ 12.500	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69 +	14	= 83	→ 11.277	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45 +	0	= 45	→ 6.114	$\boxed{3.326}$	$\boxed{7.982}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32 +	7	= 39	→ 5.299	−0.386	$\boxed{-0.980}$	$\boxed{1.090}$	42.529	→ 5.351
SSW	0.131	1 +	0	= 1	→ 0.136	0.036	3.742	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609 +	127	= 736	→ 100.000	$\boxed{4.677}$	$\boxed{8.961}$	$\boxed{0.090}$	$\boxed{222.860}$	→ 100.000

		Former official apportionment by the **D'Hondt** method								
	$100\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	11	12	13	14	15	16	17	18	19	20
SPD	28.173	170 +	37	= 207	→ 28.125	−0.351	−0.169	1.082	$\boxed{223.901}$	→ 28.173
CDU	20.679	122 +	29	= 151	→ 20.516	$\boxed{-1.199}$	−0.788	1.088	164.346	→ 20.679
GRÜNE	16.147	94 +	24	= 118	→ 16.033	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76 +	16	= 92	→ 12.500	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69 +	14	= 83	→ 11.277	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45 +	0	= 45	→ 6.114	$\boxed{3.326}$	$\boxed{7.982}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32 +	7	= 39	→ 5.299	−0.386	$\boxed{-0.980}$	$\boxed{1.090}$	42.529	→ 5.351
SSW	0.131	1 +	0	= 1	→ 0.136	0.036	3.742	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609 +	127	= 736	→ 100.000	$\boxed{4.525}$	$\boxed{8.961}$	$\boxed{0.090}$	$\boxed{222.860}$	→ 100.000

		Apportionment with minimum **absolute** deviations from party quotas								
	$100\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	21	22	23	24	25	26	27	28	29	30
SPD	28.173	170 +	37	= 207	→ 28.125	−0.351	−0.169	1.082	$\boxed{223.901}$	→ 28.173
CDU	20.679	122 +	29	= 151	→ 20.516	$\boxed{-1.199}$	−0.788	1.088	164.346	→ 20.679
GRÜNE	16.147	94 +	24	= 118	→ 16.033	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76 +	16	= 92	→ 12.500	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69 +	14	= 83	→ 11.277	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45 +	0	= 45	→ 6.114	$\boxed{3.326}$	$\boxed{7.982}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32 +	7	= 39	→ 5.299	−0.386	$\boxed{-0.980}$	$\boxed{1.090}$	42.529	→ 5.351
SSW	0.131	1 +	0	= 1	→ 0.136	0.036	3.742	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609 +	127	= 736	→ 100.000	$\boxed{4.525}$	$\boxed{8.961}$	$\boxed{0.090}$	$\boxed{222.860}$	→ 100.000

		Apportionment with minimum **relative** deviations from party quotas								
	$100\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	31	32	33	34	35	36	37	38	39	40
SPD	28.173	170 +	36	= 206	→ 27.989	$\boxed{-1.351}$	−0.651	1.087	$\boxed{223.901}$	→ 28.173
CDU	20.679	122 +	29	= 151	→ 20.516	−1.199	$\boxed{-0.788}$	$\boxed{1.088}$	164.346	→ 20.679
GRÜNE	16.147	94 +	24	= 118	→ 16.033	−0.842	−0.709	1.088	128.328	→ 16.147
FDP	12.536	76 +	16	= 92	→ 12.500	−0.267	−0.290	1.083	99.632	→ 12.536
AfD	11.320	69 +	14	= 83	→ 11.277	−0.317	−0.381	1.084	89.967	→ 11.320
CSU	5.662	45 +	0	= 45	→ 6.114	$\boxed{3.326}$	$\boxed{7.982}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32 +	8	= 40	→ 5.435	0.614	1.559	1.063	42.529	→ 5.351
SSW	0.131	1 +	0	= 1	→ 0.136	0.036	3.742	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609 +	127	= 736	→ 100.000	$\boxed{4.677}$	$\boxed{8.769}$	$\boxed{0.088}$	$\boxed{222.860}$	→ 100.000

the quotas, and the box in the bottom row shows the devoation range. Column 7 displays the vector $100(\mathbf{x}-S\mathbf{q})./(S\mathbf{q}) = 100(\mathbf{x}/S-\mathbf{q})./\mathbf{q}$ of relative deviations from the quotas in % of the quotas. Column 8 shows the vector $\mathbf{w}$ of normalized adjustment vote weights whose extreme values in boxes are inherent in the same parties as in Column 7 — in accordance with Theorem 1. Columns 9–10 prove that the adjustment vote weights make the factions' vote powers $\mathbf{x}.*\mathbf{w}$ equal to the faction's quotas in Column 1, i.e. $100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}} = 100\mathbf{q}$.

At this point, one can ask oneself whether the official Sainte-Laguë apportionment of the Bundestag with 736 seats is really optimal? The answer is given in the third and fourth blocks of Table 6, which specify the Bundestag's optimal apportionments with minimum *absolute* and minimum *relative* deviations of factions from quotas, respectively. The three unadjusted CSU's overhangs imply the highest and equal for all the methods positive deviation of the CSU faction from its quota by 3.326 seats, whereas the negative deviations are different, indicating the different accuracy in adjusting other party factions to their quotas. The optimal apportionment for the absolute accuracy criterion in the third block differs from the official one in the seats of SPD and CDU (cf. Columns 24 and 4). This implies a smaller range of absolute deviations from quotas in Column 26 versus Column 6 (4.525 seats versus 4.677 seats), meaning a better fit of factions to quotas. The optimal apportionment for the relative accuracy criterion in the fourth block differs from the official one in the seats of SPD and LINKE (cf. Columns 34 and 4). This implies a smaller range of relative deviations of factions from quotas in Column 37 compared with that in Column 7 (8.769% versus 8.961%) and, correspondingly, a smaller range of adjustment vote weights in Column 38 compared with that in Column 8 (0.088 versus 0.090), meaning that the party factions can be adjusted better under the relative accuracy criterion as well. We conclude that the official Sainte-Laguë method is not optimal — meaning that it is not most fair.

The second block of Table 6 confirms the observation that the D'Hondt method slightly favors big winners. Indeed, the SPD gets 207 seats in Column 14, by one more than under the Sainte-Laguë method in Column 4. Here, the D'Hondt apportionment is optimal with respect to the absolute accuracy criterion, but as we see later it is not always the case.

4.4 Using adjustment vote weights to reduce the Bundestag size

As already mentioned, adjustment vote weights solve most apportionment problems, at least formally. Regardless of the apportionment, adjustment vote weights make the factions' voting powers proportional to the party quotas. In other words, the Bundestag can be downsized and the faction size ratio violated without affecting the factions' voting powers.

The critical question is however to which degree the Bundestag members can differ in their vote weights, i.e. to which degree the inequality of Bundestag members is acceptable. The range of vote weights should probably be limited so as not to make some members of the Bundestag too privileged and others too discriminated against. This ethical question belongs to the competence of politicians and philosophers, being beyond the scope of our study. We only illustrate this idea using the example of the Bundestag with 630 seats — as proposed by the 12th President of the Bundestag (2005–2017) Norbert Lammert [Roßner 2016].

Table 7 shows the apportionments of the hypothetical 2021 Bundestag with 630 seats obtained by the Sainte-Laguë and D'Hondt methods and two optimization models. In this case, the Sainte-Laguë apportionment coincides with the D'Hondt apportionment, but both are not optimal from the viewpoint of the absolute or the relative accuracy criteria. In all the four apportionments, the maximum absolute deviation from quotas is due to the CSU with most overhangs, which is as high as 9.328 seats (boxed in Columns 6, 16, 26 and 36). The non-optimality of the Sainte-Laguë and the D'Hondt apportionments manifests itself in even greater negative deviations of certain factions from their quotas, which exacerbates their disadvantageous position.

The adjustment vote weights adjust the factions' vote powers to quotas but introduce a certain inequality in individual vote powers of the Bundestag members. By virtue of Theorem 1, this inequality

Table 7: Apportionments of the 2021 Bundestag with 630 seats (as proposed by Norbert Lammert)

		Official apportionment by the **Sainte-Laguë** method								
	Party quota in Bundestag, in % $100\,\mathbf{q}$	Minimum seat entitlement $\mathbf{m}$	Adjustment seats $\mathbf{x}-\mathbf{m}$	Seats (apportionment) $\mathbf{x}$	Seats, in % of the Bundestag seats $100\frac{\mathbf{x}}{S}$	Absolute deviation from the quota, in number of seats $\mathbf{x}-S\mathbf{q}$	Relative deviation from the quota, in % of the quota $100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	Adjustment vote weight $\mathbf{w}$	Party's votes (Seats × Vote weight) $\mathbf{x}.*\mathbf{w}$	Party's share of power, in % $100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	1	2	3	4	5	6	7	8	9	10
SPD	28.173	170	+ 5	= 175	→ 27.778	$\boxed{-2.488}$	−1.402	1.279	$\boxed{223.901}$	→ 28.173
CDU	20.679	122	+ 6	= 128	→ 20.317	−2.279	−1.749	1.284	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 2	= 78	→ 12.381	−0.979	−1.239	1.277	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	$\boxed{9.328}$	$\boxed{26.150}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32	+ 1	= 33	→ 5.238	−0.713	$\boxed{-2.116}$	$\boxed{1.289}$	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.175	21.198	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609	+21	= 630	→100.000	$\boxed{11.816}$	$\boxed{28.266}$	$\boxed{0.289}$	$\boxed{222.860}$	→100.000

		Former official apportionment by the **D'Hondt** method								
	$100\,\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	11	12	13	14	15	16	17	18	19	20
SPD	28.173	170	+ 5	= 175	→ 27.778	$\boxed{-2.488}$	−1.402	1.279	$\boxed{223.901}$	→ 28.173
CDU	20.679	122	+ 6	= 128	→ 20.317	−2.279	−1.749	1.284	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 2	= 78	→ 12.381	−0.979	−1.239	1.277	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	$\boxed{9.328}$	$\boxed{26.150}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32	+ 1	= 33	→ 5.238	−0.713	$\boxed{-2.116}$	$\boxed{1.289}$	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.175	21.198	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609	+21	= 630	→100.000	$\boxed{11.816}$	$\boxed{28.266}$	$\boxed{0.289}$	$\boxed{222.860}$	→100.000

		Apportionment with minimum **absolute** deviations from party quotas								
	$100\,\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	21	22	23	24	25	26	27	28	29	30
SPD	28.173	170	+ 6	= 176	→ 27.937	−1.488	−0.838	1.272	$\boxed{223.901}$	→ 28.173
CDU	20.679	122	+ 7	= 129	→ 20.476	−1.279	−0.982	1.274	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 1	= 77	→ 12.222	$\boxed{-1.979}$	−2.506	1.294	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	$\boxed{9.328}$	$\boxed{26.150}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32	+ 0	= 32	→ 5.079	−1.713	$\boxed{-5.082}$	$\boxed{1.329}$	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.175	21.198	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609	+21	= 630	→100.000	$\boxed{11.307}$	$\boxed{31.232}$	$\boxed{0.329}$	$\boxed{222.860}$	→100.000

		Apportionment with minimum **relative** deviations from party quotas								
	$100\,\mathbf{q}$	$\mathbf{m}$	$\mathbf{x}-\mathbf{m}$	$\mathbf{x}$	$100\frac{\mathbf{x}}{S}$	$\mathbf{x}-S\mathbf{q}$	$100\left(\frac{\mathbf{x}}{S}-\mathbf{q}\right)./\mathbf{q}$	$\mathbf{w}$	$\mathbf{x}.*\mathbf{w}$	$100\frac{\mathbf{x}.*\mathbf{w}}{\mathbf{x}'\mathbf{w}}$
	31	32	33	34	35	36	37	38	39	40
SPD	28.173	170	+ 4	= 174	→ 27.619	$\boxed{-3.488}$	$\boxed{-1.965}$	$\boxed{1.287}$	$\boxed{223.901}$	→ 28.173
CDU	20.679	122	+ 6	= 128	→ 20.317	−2.279	−1.749	1.284	164.346	→ 20.679
GRÜNE	16.147	94	+ 6	= 100	→ 15.873	−1.726	−1.697	1.283	128.328	→ 16.147
FDP	12.536	76	+ 2	= 78	→ 12.381	−0.979	−1.239	1.277	99.632	→ 12.536
AfD	11.320	69	+ 1	= 70	→ 11.111	−1.318	−1.848	1.285	89.967	→ 11.320
CSU	5.662	45	+ 0	= 45	→ 7.143	$\boxed{9.328}$	$\boxed{26.150}$	$\boxed{1.000}$	45.000	→ 5.662
LINKE	5.351	32	+ 2	= 34	→ 5.397	0.287	0.850	1.251	42.529	→ 5.351
SSW	0.131	1	+ 0	= 1	→ 0.159	0.175	21.198	1.041	$\boxed{1.041}$	→ 0.131
Sum / $\boxed{\text{Range}}$	100.000	609	+21	= 630	→100.000	$\boxed{12.816}$	$\boxed{28.115}$	$\boxed{0.287}$	$\boxed{222.860}$	→100.000

achieves its minimum (corresponding to the smallest range of adjustment vote weights) when the *relative* deviations of factions from their quotas are minimized. Indeed, the optimal apportionment for the relative accuracy criterion in Column 34 of Table 7 implies the range of adjustment vote weights as small as 0.287 boxed at the bottom of Column 38 (cf. with 0.329 in Column 28 and 0.289 in Columns 8 and 18). In other words, the vote power inequality of the Bundestag members can be kept within 28.7%.

It should be emphasized that avoiding oversized assemblies leads to unadjusted overhangs. In turn, unadjusted overhangs inevitably lead to the alternative: (1) infringement of collective powers of factions due to quota violations against (2) infringement of individual powers of deputies due to unequal adjustment vote weights. For the Bundestag with 736 members including three tolerated overhangs, this alternative, even optimized, looks as follows: (1) the infringement of collective powers of factions with the range of quota violations by more than 4.5 seats (see the bottom of Column 26 of Table 6) against (2) the infringement of individual powers of deputies with the range of adjustment vote weights of 0.088 (see the bottom of Column 38 of Table 6). For the Bundestag with 630 members including nine tolerated overhangs, the optimized alternative is: (1) the infringement of collective powers of factions with the range of quota violations by more than 11.3 seats (see the bottom of Column 26 of Table 7) against (2) the infringement of individual powers of deputies with the range of adjustment vote weights of 0.287 (see the bottom of Column 38 of Table 7).

In summary, even in the 630-seat Bundestag, the right balance of power between the parties is feasible if adjusted vote weights were used. However, the question remains open as to whether the degree of inequality of 29% of the Bundestag members is acceptable or not.

5 The D'Hondt and Sainte-Laguë methods vs optimization

Thus, neither the Sainte-Laguë nor the D'Hondt apportionment method is optimal. Let us see how the performance of the two heuristic methods relate to the two optimization methods considered so far. For this purpose, we track some characteristics of the Bundestag apportionments $\mathbf{x} = f(\mathbf{q}, \mathbf{m}, \text{Method}, S)$ for the actual 2021 Bundestag quotas $\mathbf{q}$, the actual 2021 minimum Bundestag seats $\mathbf{m}$ and without minimum seats ($\mathbf{m} = \mathbf{0}$) and computed by four Methods for variable Bundestag sizes S.

5.1 The case with minimum seats reserved for the parties

Figure 1 illustrates the performance of the four apportionment methods from the viewpoint of criterion of absolute accuracy in fitting factions to quotas. Each pair of curves of the same color limits the positive and negative deviations of factions from quotas for one apportionment method.[11] More precisely, we plot functions

$$\left.\begin{array}{l}\max[\mathbf{x} - S\mathbf{q}]\\ \min[\mathbf{x} - S\mathbf{q}]\end{array}\right\}, \quad \text{where} \quad \mathbf{x} = f(\mathbf{q}, \mathbf{m}, \text{Method}, S), \quad \text{Method} = \text{SL, DH, AA, RA}, \quad S = 630, \ldots, 830 .$$

For the starting points of the curves, see the boxed figures in Columns 6, 16, 26 and 36 in Table 7.

By virtue of (11), the CSU overhangs remain unadjusted for the Bundestag sizes $S < 786$, determining the maximum positive deviations of factions from quotas. Respectively, the four upper curves in Figure 1, which depict these deviations, coincide. As the Bundestag grows, the CSU overhangs are gradually adjusted, and the four curves monotonically decrease unless they enter the yellow zone of the apportionment accuracy of 0.5 seat. For the size of the Bundestag $S \geq 786$, the faction size rounding to integers causes uneven leaps of the curves resulting in their sawtooth shape.

The lower four curves, which show the greatest negative deviations of party factions from quotas, are predetermined by vector $\mathbf{m}$, which specifies the allocation of totally 609 minimum seats reserved for

[11] Since the curves associated with different apportionment methods partially coincide, they are slightly shifted vertically to make them easier to distinguish.

the parties. When the Bundestag size S is close to 630, the adjustment seats, which complement 609 minimum seats, are too few to adjust heavily underrepresented factions. As the Bundestag grows, the party quotas are filled and the negative deviations of factions from quotas become less significant, which is reflected by the growth trend of the lower curves. Their non-monotonicity is caused by allocating the next adjustment seat to different parties, which changes the faction ratio at each step, resulting in even leaps of the curves. It is noteworthy that the upper and lower curves enter the yellow zone simultaneously. This is explained by the fact that the unadjusted overhangs of the CSU imply underrepresentation of some other parties. Therefore, for the minimum negative deviations, the rounding-related apportionment accuracy of 0.5 seat can only be achieved if all overhangs are adjusted.

As expected, the smallest deviations of factions from quotas shown by the black curves are achieved by optimization with the absolute accuracy criterion. On the contrary, the greatest deviations of factions from quotas shown by blue curves result from optimization with the relative accuracy criterion. As long as there are unadjusted overhangs ($S < 786$), the performance of the D'Hondt method (characterized by the red curves) looks better than that of the Sainte-Laguë method (characterized by the green curves), whereas for the Bundestag with adjusted overhangs ($S \geq 786$), the Sainte-Laguë method becomes quasi-optimal with respect to the absolute accuracy criterion (the green curves coincide with the black curves) but the D'Hondt method performs 'as badly' as the relative accuracy optimization method or even worse (the red curves mostly coincide with the blue ones or even surpass their limits).

Figure 2 illustrates the performance of the four apportionment methods from the viewpoint of criterion of relative accuracy in fitting factions to quotas. It shows four curves that track the range of the (normalized) adjustment vote weights $\mathbf{w} = \dfrac{S\mathbf{q}./\mathbf{x}}{\min[\mathbf{q}./\mathbf{x}]}$ for apportionments computed by four models SL, DH, AA, RA and variable Bundestag size:

$$\text{range}\left[\frac{\mathbf{q}./\mathbf{x}}{\min[S\mathbf{q}./\mathbf{x}]}\right], \text{ where } \mathbf{x} = f(\mathbf{q}, \mathbf{m}, \text{Method}, S), \text{ Method} = \text{SL, DH, AA, RA}, \quad S = 630, \ldots, 830.$$

For the starting points of the four curves, see the boxed figures at the bottom of Columns 8, 18, 28 and 38 in Table 7.

As the Bundestag starts to grow, the apportionments obtained by the four methods are becoming more accurate. Since the adjustment vote weights compensate the inaccuracies in fitting factions to quotas, the more accurate the apportionment, the smaller the range of the adjustment vote weights. Respectively, the curves computed for four apportionment methods tend to decrease, though not synchronously. The lower blue curve plots the range of the adjustment vote weights computed by the optimization model with the criterion of minimum relative accuracy, which always guarantees the minimum range — exactly in agreement with Theorem 1. One can say that the optimization with the relative accuracy criterion takes precedence if adjustment vote weights are applied. The upper black curve shows that the greatest range of adjustment vote weights is inherent in the apportionments optimized under the criterion of minimum absolute accuracy. The red and green curves for the D'Hondt and Sainte-Laguë methods, respectively, are nested below.

The four curves reach their minimum of ca. 0.041 and remain at this level for the Bundestag sizes $769 \leq S \leq 790$. This is explained by the fact that, for these Bundestag sizes, the CSU with its 45 minimum seats is the most overrepresented party, and the SSW with its single seat is the most underrepresented one, whence their *non-normalized* adjustment vote weights $w_{\text{CSU}}(S)$ and $w_{\text{SSW}}(S)$ are the smallest and the greatest, respectively. As follows from (9),

$$w_{\text{CSU}}(S) = S q_{\text{CSU}}/45 , \qquad w_{\text{SSW}}(S) = S q_{\text{SSW}}/1 .$$

Substituting the values of the Bundestag quotas from Table 1, we obtain the ratio

$$\frac{w_{\text{SSW}}(S)}{w_{\text{CSU}}(S)} = \frac{S q_{\text{SSW}}}{S q_{\text{CSU}}/45} = \frac{45 \cdot 0.00131}{0.05662} \approx 1.041 \quad \text{for all} \quad S = 769, \ldots, 790 . \tag{12}$$

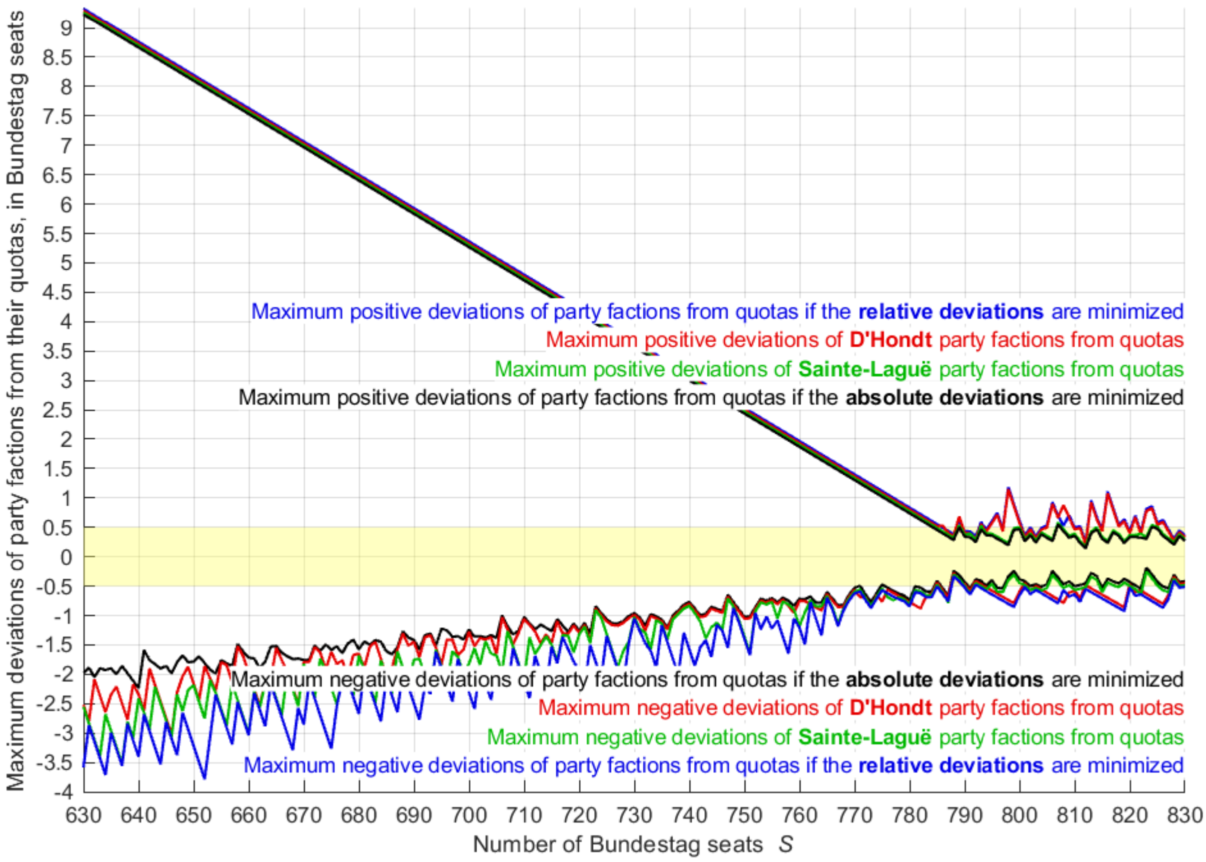


Figure 1: Maximum deviations of party factions *with minimum seats* $\mathbf{m}$ from their quotas $\mathbf{q}$. The yellow zone shows the range of accuracy of fitting factions to quotas within 0.5 seat.

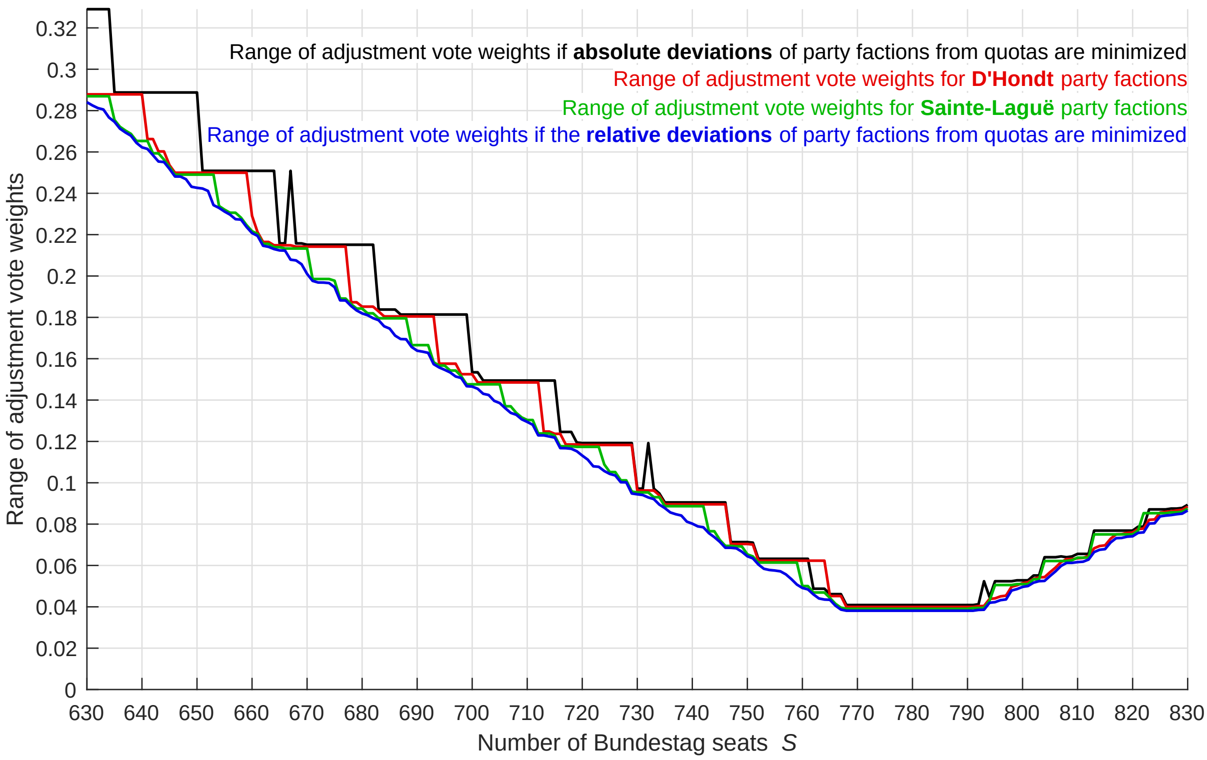


Figure 2: Range of adjustment vote weights $\mathbf{w}$ for members of party factions *with minimum seats* $\mathbf{m}$.

Hence, for every S from this interval, the normalization (10) transforms the CSU's and SSW's adjustment vote weights to 1 and 1.041, respectively, resulting in the constant range of the normalized adjustment vote weights = 0.041.

As the Bundestag size reaches $S = 791$, the CSU yields the role of the most underrepresented party to the AfD, at $S = 793$ it is the SPD, and since there are no longer overhangs this role goes from one party to another. Due to a very small quota, the SSW faction long remains with a single member as the Bundestag grows, being more and more underrepresented. Its adjustment vote weight grows correspondingly increasing the range of the normalized adjustment vote weights, which is reflected in Figure 2.

Like in Figure 1, the transition from presence to absence of overhangs (at the Bundestag size of ca. $S = 790$) reverts the vertical order of the Sainte-Laguë and D'Hondt curves. As long as there are unadjusted overhangs, the performance of the Sainte-Laguë method looks better than that of the D'Hondt method (the green curve is closer to the lower blue one than the red curve). After the overhangs have been levelled off, the green Sainte-Laguë curve joins the upper black curve of the optimization method with the absolute accuracy criterion, and the red D'Hondt curve joins the lower blue curve of the optimization method with the relative accuracy criterion. Thus, from the viewpoint of the relative accuracy criterion (= minimum inequality between adjustment vote weights) the Sainte-Laguë method better performs in the presence of overhangs, whereas in their absence the D'Hondt method looks preferable, being quasi optimal.

5.2 The case with no seats reserved for the parties

Let us see whether the D'Hondt method in the absence of overhangs is really better than the Sainte-Laguë method from the viewpoint of the relative accuracy criterion. For this purpose, we assume no minimum seats reserved for the parties, i.e. $\mathbf{m} = \mathbf{0}$. The results are visualized in Figures 3 and 4 whose layout and scales are the same as in Figures 1 and 2. The D'Hondt red curve in Figure 4 joins the lower blue curve of the relative optimization model but it appears only for the Bundestag size $S = 760$ — when the D'Hondt method allocates the first seat to the minor party SSW with its quota of 0.131%. In fact, applying (9) to the SSW faction with no members $x_{\text{SSW}} = 0$ would give the infinite adjustment vote weight $w_{\text{SSW}} = \infty$, which obviously makes no sense. Here, we observe the D'Hondt method's particularity of disadvantaging minor parties who 'come into play' at a rather late stage.[12]

This particularity is also evident in Figure 3. Allocating no seat to the SSW faction results in its negative deviation from the quota of about one seat, making the required absolute apportionment accuracy of 0.5 seat unattainable up to the Bundestag size of 760 members. Moreover, as long as the SSW quota is not filled the positive deviations of factions from quotas are also greater than that obtained using the relative optimization model (the upper red D'Hondt curve is significantly above the blue curve of the relative optimization model).

Thus, the Sainte-Laguë method generally performs better than the D'Hondt method, being however not optimal. The D'Hondt method is definitively worse if minor parties have no minimum seats. In extreme cases it can even fail to reach the required accuracy of 0.5 seat.

Another issue is that both methods change their performance as all overhangs have been adjusted. As long as overhangs are not levelled out, the D'Hondt method is closer to the optimization with the absolute accuracy criterion, and after that it performs mostly as the optimization model with the relative accuracy criterion. The Sainte-Laguë method performs in the opposite way. In the presence of overhangs, it is closer to the optimization with the relative accuracy criterion, and as they have been adjusted — it performs as the optimization model with the absolute accuracy criterion. In this respect, the Saite-Laguë and the d'Hondt methods are somewhat short of optimization consistency.

[12]The lack of a flat segment at the minimum of the four curves is explained by the lack of overhangs that create the effect analyzed in (12).

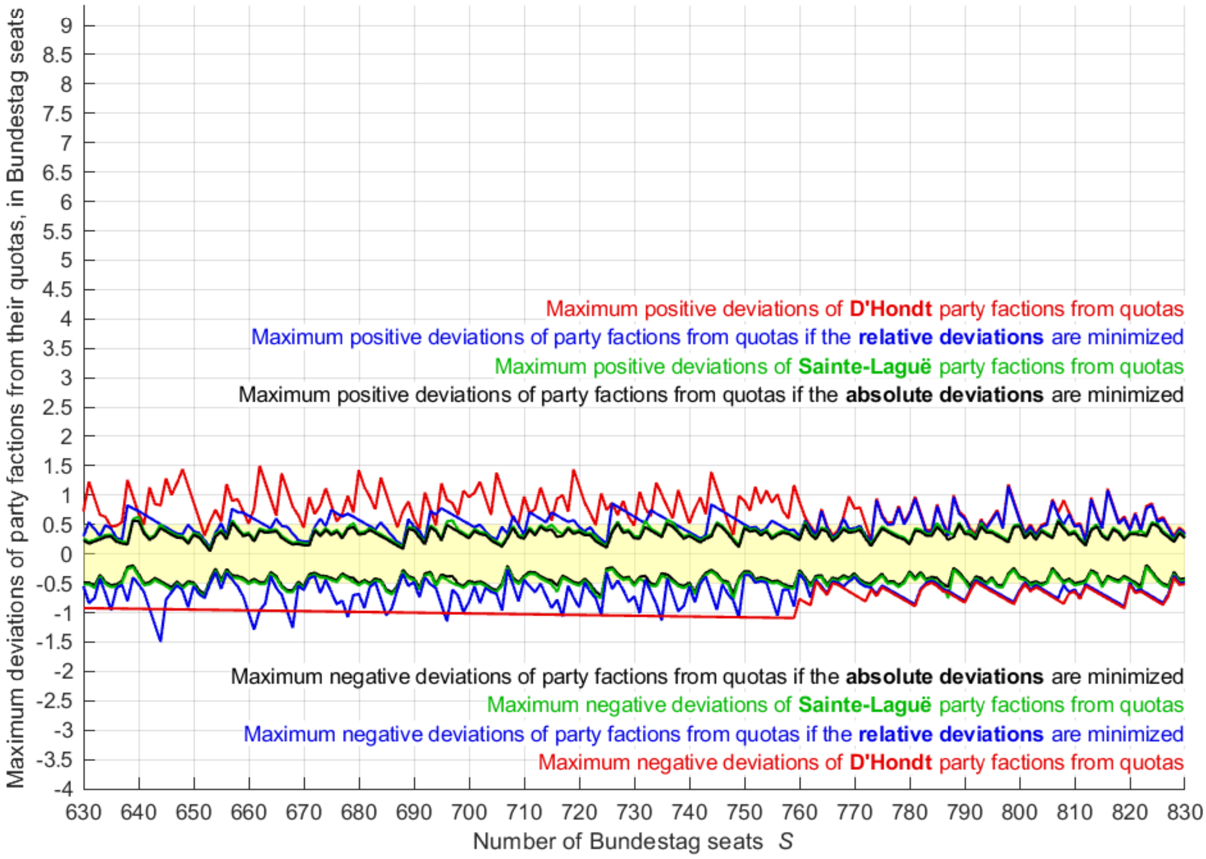


Figure 3: Maximum deviations of party factions *with no minimum seats* $(\mathbf{m} = \mathbf{0})$ from their quotas $\mathbf{q}$. The yellow zone shows the range of accuracy of fitting factions to quotas within 0.5 seat.

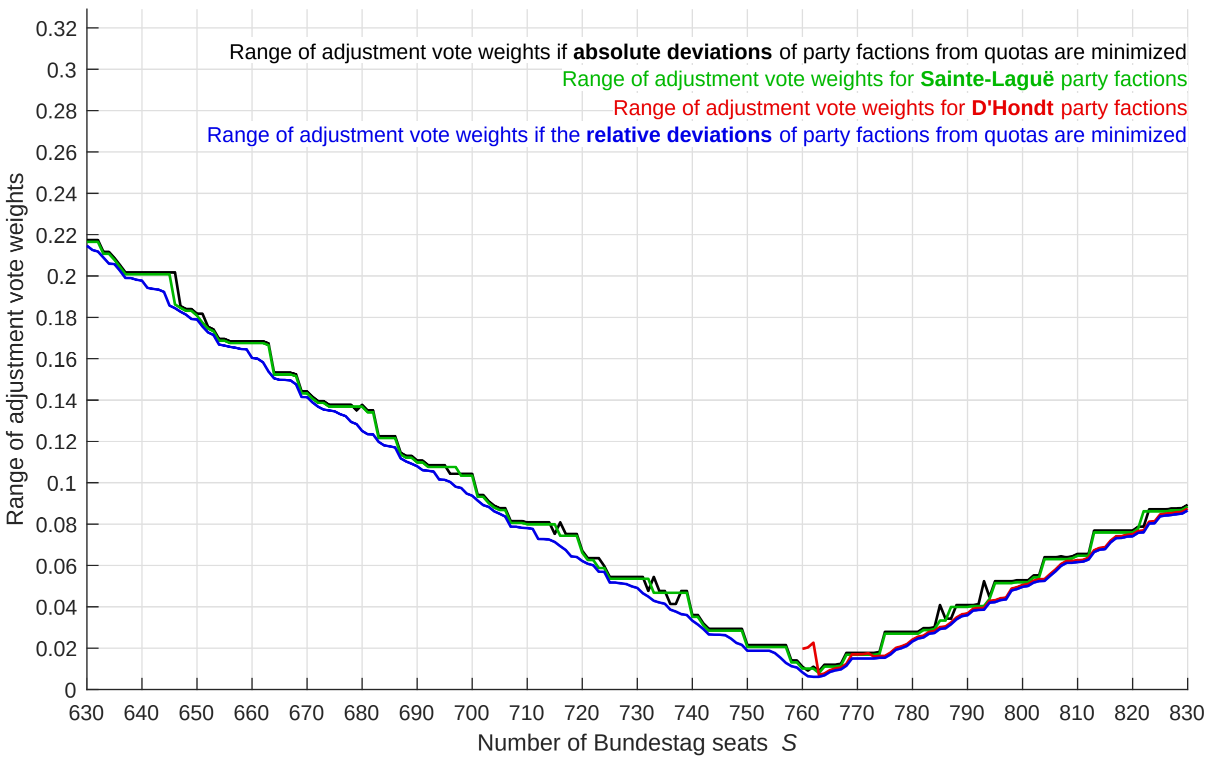


Figure 4: Range of adjustment vote weights $\mathbf{w}$ for members of party factions *with no minimum seats* $(\mathbf{m} = \mathbf{0})$.

6 Conclusions

The heuristic apportionment methods of D'Hondt and Sainte-Laguë are not as perfect as it is commonly believed. They find acceptable solutions but not necessarily the best ones. Moreover, they are not quite consistent regarding absolute and relative apportionment accuracy criteria. We show that better (= more fair) and logically more consistent apportionments can be found using rigorous discrete optimization techniques. The second point is the vicious practice of obtaining an apportionment with overhangs in two steps: first apportion fewer seats with no unadjusted overhangs and then add the missing ones. We show that this task should be done in one run using optimality criteria subject to constraints.

Oversized assemblies can be avoided by introducing adjustment vote weights that extend the adjustment practices used in apportionment anyway. This device can reduce the assembly size on the one hand, and, on the other hand, make the faction' voting powers exactly proportional to the votes the party receive in elections. The following alternative also speaks in favor of adjustment vote weights: in the presence of unadjusted overhangs (1) either infringement of factions' collective powers due to quota violations or (2) infringement of deputies' individual powers due to the inequality of adjustment vote weights.

The replacement of intuitively natural D'Hondt and Sainte-Laguë methods by less evident optimization models meets the actual technical possibilities and digitalization trends in society and government. The only question is whether to follow the principle of 'one man, one vote' and then adjust overhangs by numerous leveling seats — in which case the absolute accuracy criterion is relevant — or to withdraw from this principle, limit the parliament size and introduce adjustment vote weights — then the relative accuracy criterion should be applied.

References

[Baixauli-Soler Alfaro-Cid and Fernandez-Blanco 2011] Baixauli-Soler JS, Alfaro-Cid E, Fernandez-Blanco MO (2011) Mean-VaR portfolio selection under real constraints. Computational Economics 37 (2): 113–131. DOI: doi.org/10.1007/s10614-009-9195-1.

[Balinski and Young 1979] Balinski M, Young HP (1979) Criteria for proportional representation. Operations Research 27(1): 80–95

[Balinski and Young 1982] Balinski M, Young HP (1982) Fair representation: meeting the ideal of one man, one vote. Yale University Press, New Haven. 2nd ed.: Brookings Institution Press, Washington DC, 2001

[Bundestag 2023] Bundestag (2023) Wikimedia Foundation, last modified 20:33, 24 March 2023 https://en.wikipedia.org/wiki/Bundestag.

[Bundeswahlleiter 2021] Bundeswahlleiter (2021) Wahl zum 20. Deutschen Bundestag am 26. September 2021. Heft 3. Endgltige Ergebnisse nach Wahlkreisen. [Election of the 20th German Bundestag on September 26, 2021. Vol 3: Final results by constituencies]. Bundeswahllwiter, Wiesbaden. https://www.bundeswahlleiter.de/dam/jcr/cbceef6c-19ec-437b-a894-3611be8ae886/btw21_heft3.pdf. Cited 8 April 2023

[D'Hondt 1882] D'Hondt V (1882) Système pratique et raisonné de représentation proportionelle. Librairie C Muq quardt, Merzbach et Falk, éditeurs, libraires du roi et du cte de Flandre, Bruxelles

[D'Hondt 1885] D'Hondt V (1885) Exposé du système pratique de représentation proportionnelle. Gante, Imprimerie Eug. Van der Haeghen.

[D'Hondt method 2023] D'Hondt method (2023). Wikimedia Foundation, last modified 17:43, 26 February 2023 https://en.wikipedia.org/wiki/D'Hondt_method. Cited 3 March 2023

[D'Hondt-Verfahren 2023] D'Hondt-Verfahren (2021). Wikimedia Foundation, last modified 17:43, 3 March 2023 https://de.m.wikipedia.org/wiki/D'Hondt-Verfahren. Cited 12 Aug 2021

[Deutscher Bundestag 2023] Deutscher Bundestag (2023) Sitzverteilung des 20. Deutschen Bundestages. https://www.bundestag.de/parlament/plenum/sitzverteilung_20wp Retrieved on April 5, 2023.

[Dorfleitner and Klein 1999] Dorfleitner G, Klein T (1999) Rounding with multiplier methods: an efficient algorithm and applications in statistics. Statistical Papers 40, 143157. DOI: doi.org/10.1007/BF02925514

[Edelman et al. 2014] Edelman PH, Thomas RS, Thompson RB (2014) Shareholder voting in an age of intermediary capitalism. Southern California Law Review, 87: 1359–1434

[Finthammer 2018] Finthammer V (2018) Bericht des Rechnungshofs: Bundestag wird immer größer und teuer. Deutschlandfunk, 8 Oct 2018. https://www.deutschlandfunk.de/bericht-des-rechnungshof-wird-immergroesser-und.1766.de.html?dram:article_id=429962. Cited 13 Oct 2021

[Freiberger and Thomas 2019] Marianne Freiberger M, Thomas R (2019) Understanding Numbers: Simplify life's mathematics. Decode the world around you.: 20 thought-provoking lessons (BUILD+BECOME). London, White Lion Publishing. The relevant chapter summary: https://plus.maths.org/content/maths-minute-dhondt-method.

[Herrero and Martínez 2008] Herrero C, Martínez (2008) Balanced allocation methods for claims problems with indivisibilities. Social Choice and Welfare 30 (4): 603–617. DOI: doi.org/10.1007/s00355-007-0262-z

[Holler and Nurmi 2013] Holler MJ, Nurmi H (2013) (Eds) Power, Voting and Voting Power: 30 Years after. Springer, Berlin–Heidelberg

[Kruse et al. 2016] Kruse R, Borgelt Ch, Braune Ch, Mostaghim S, Steinbrecher M (2016) Computational Intelligence: A Methodological Introduction. 2nd ed. New York, Springer. DOI: doi.org/10.1007/978-1-4471-5013-8

[Kubota and Aritsugi 2017] Kubota T, Aritsugi M (2017) Assignment strategies for ground truths in the crowdsourcing of labeling tasks. Journal of Systems and Software 126, 113–126. DOI: doi.org/10.1016/j.jss.2016.06.061.

[Lijphart 2003] Lijphart A (2003) Degrees of proportionality of proportional representation formulas. In: Grofman B, Lijphart A (eds) Electoral laws and their political consequences. Algora Publishing, New York: 170–179

[Mazurkiewicz and Mercik 2005] Mazurkiewicz M, Mercik JW (2005) Modified Shapley-Shubik power index for parliamentary coalitions. Badania Operacyjene i Decyzje, 2: 43–52

[Medzihorsky 2019] Medzihorsky J (2019) Rethinking the D'Hondt method. Political Research Exchange, an ECPR journal 1(1), 1-15. DOI: doi.org/10.1080/2474736X.2019.1625712.

[Mixed-member proportional representation 2023] Mixed-member proportional representation (2023) Wikimedia Foundation, last modified 21 February 2023 04:05. https://en.m.wikipedia.org/wiki/Mixed-member_proportional_representation.

[O'Leary Grofman and Elklit 2005] O'Leary B, Grofman B, Elklit J (2005) Divisor methods for sequential portfolio allocation in multi-party executive bodies: Evidence from Northern Ireland and Denmark. American Journal of Political Science 49 (1): 198–211.

[Pukelsheim 2002] Pukelsheim F (2002) Die Väter der Mandatszuteilungsverfahren. Spectrum der Wissenschaft, Sep 2002: 83. Internet version: Die Drei in Deutschland verwendeten Mandatszuteilungsmethoden und ihre Namenspatrone. math.uni-augsburg.de/htdocs/emeriti/pukelsheim/2002g.html. Cited 14 Oct 2021

[Pukelsheim 2007] Pukelsheim F (2007) Seat bias formulas in proportional representation systems. Paper at the ECPR General Conference, Pisa, September 6–8, 2007

[Results of the 2021 German federal election 2023] Results of the 2021 German federal election (2023) Wikimedia Foundation, last modified 08:40, 25 March 2023. https://en.wikipedia.org/wiki/Results_of_the_2021_German_federal_election

[Roßner 2016] Roßner S (2016) Die Grenten des Wachstums. Legal Tribune Online 02.11.2016 ito.de/recht/hintergruende/h/bundestag-mitglied-begrenzung-wahlrecht-landeslisten-zweitstimmen/. Cited 12 Oct 2021

[Sainte-Laguë 1910] Sainte-Laguë A (1910) La représentation proportionelle et la méthode des moindres carrées. Annales scientifiques de l'École normale superièure 27: 529–542. Abstract in: Lijphart A, Gibberd RW (1977) Thresholds and payoffs in list systems of proportional representation, Appendix 2. Eur J Pol Res 5: 219–244

[Sainte-Laguë method 2023] Sainte-Laguë method (2023) Wikimedia Foundation, last modified 18:30, 26 February 2023?.? https://en.wikipedia.org/wiki/Sainte-Lagu%C3%AB_method. Cited 3 March 2023

[Sainte-Laguë-Verfahren 2023] Sainte-Laguë-Verfahren (2023) Wikimedia Foundation, last modified 10. Februar 2023 11:19. https://de.m.wikipedia.org/wiki/Sainte-Lagu%C3%AB-Verfahren.

[Shapley and Shubik 1954] Shapley LS, Shubik M (1954) A method for evaluating the distribution of power in a committee system. American Political Science Review, 48(3): 787–792

[Spiegel online 2019] Spiegel online (2019, 20 Sep) Staatsrechtler halten Bundestag für zu groß. spiegel.de/politik/deutschland/bundestag-staatsrechtler-halten-709-abgeordnete-fuer-zu-viel-a-1287739.html. Cited 12 Oct 2021

[Takai 2016] Takai K (2016) CodHonEditor: spreadsheets for codon optimization and editing of protein coding sequences. Nucleosides, Nucleotides and Nucleic Acids 35 (5): 223–232. DOI: doi.org/10.1080/15257770.2015.1127962

[Tangian 2022] Tangian A (2022) Analysis of the 2021 Bundestag elections. 3/4. Tackling the Bundestag growth. ECON Working paper 153 (January 2022). Karlsruhe Institute of Technology, Karlsruhe. DOI doi.org/10.5445/IR/1000143158

[Varela and Prado-Dominguez 2012] Varela D, Prado-Dominguez J (2012) Negotiating the Lisbon Treaty: redistribution, efficiency and power indices. Czech Economic Review, 6(2): 107–124.

[Zeit online 2019] Zeit online (2019, 21 Sep) Thomas Oppermann will Wahlrechtsreform noch dieses Jahr. zeit.de/politik/deutschland/2019-09/bundestag-thomas-oppermann-einigung-wahlrechtsreform-verkleinerung?print. Cited 12 Oct 2021

[Zhang et al. 2016] Zhang B, Zhang Zh, Ren Z, Ma J, Wang W (2016) Energy-efficient software-defined data collection by participatory sensing. IEEE Sensors Journal 16 (20): 7315–7324.

Working Paper Series in Economics

recent issues

No. 161 *Andranik S. Tangian:* Apportionment in times of digitalization, April 2023

No. 160 *Klaus Nehring, Clemens Puppe:* Multi-dimensional social choice under frugal information: the Tukey median as Condorcet winner ex ante, March 2023

No. 159 *Clemens Puppe, Arkadii Slinko:* Maximal Condorcet domains. A further progress report, December 2022

No. 158 *Daniel Hoang, Kevin Wiegratz:* Machine learning methods in finance: recent applications and prospects, December 2022

No. 157 *David Ehrlich, Nora Szech:* How to start a grassroots movement, September 2022

No. 156 *Klaus Nehring, Clemens Puppe:* Condorcet solutions in frugal models of budget allocation, March 2022

No. 155 *Clemens Puppe, Jana Rollmann:* Participation in voting over budget allocations. A field experiment, February 2022

No. 154 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 4/4. The third vote application, January 2022

No. 153 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 3/4. Tackling the Bundestag growth, January 2022

No. 152 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 2/4. Political spectrum, January 2022

No. 151 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 1/4. Representativeness of the parties and the Bundestag, January 2022