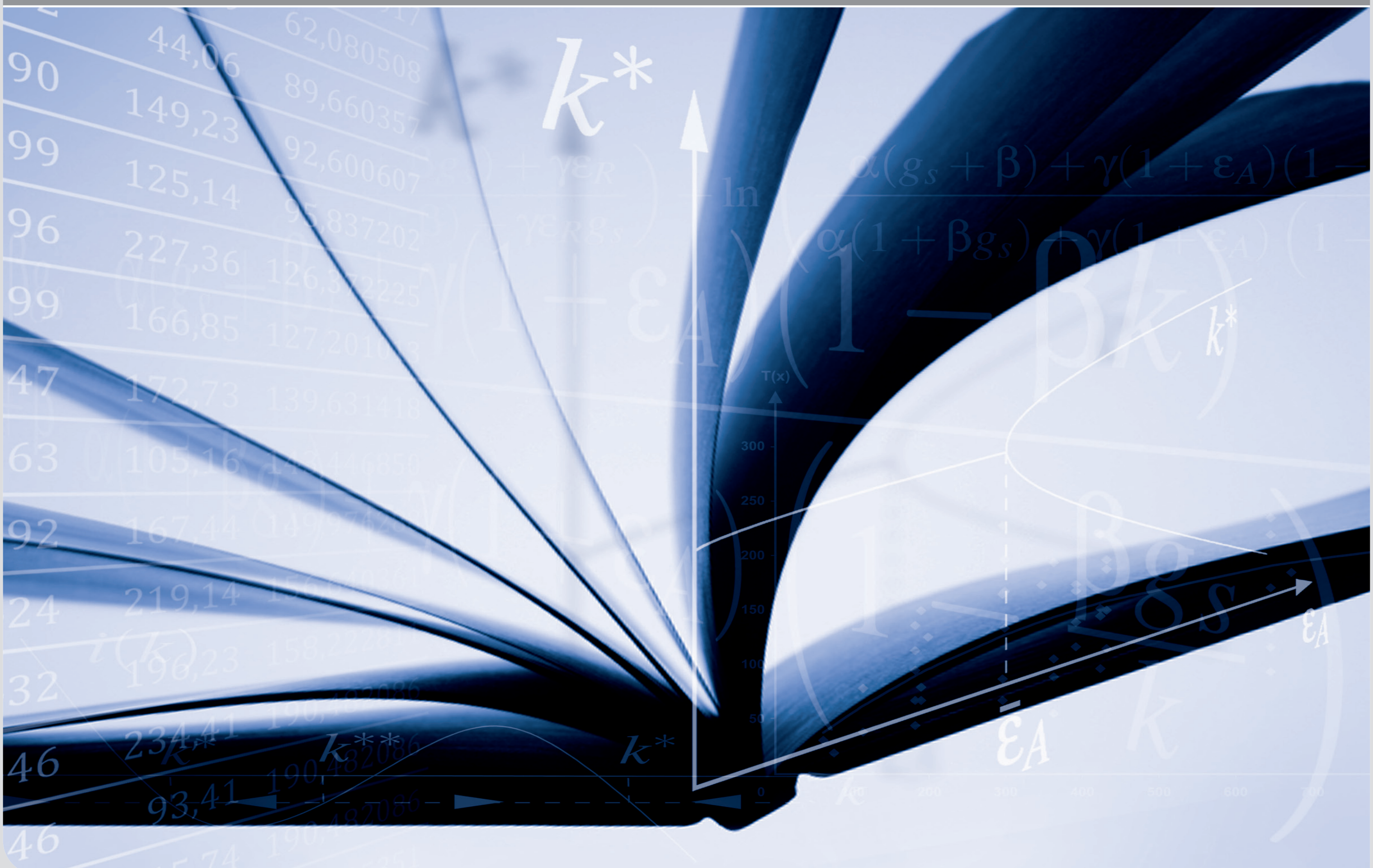
Maximal Condorcet domains. A further progress report

by Clemens Puppe, Arkadii Slinko

No. 159 | DECEMBER 2022

WORKING PAPER SERIES IN ECONOMICS



KIT – Universität des Landes Baden-Württemberg und
nationales Forschungszentrum in der Helmholtz-Gemeinschaft

econpapers.wiwi.kit.edu

Impressum

Karlsruher Institut für Technologie (KIT)
Fakultät für Wirtschaftswissenschaften
Institut für Volkswirtschaftslehre (ECON)

Kaiserstraße 12
76131 Karlsruhe

KIT – Die Forschungsuniversität in der Helmholtz-Gemeinschaft

Working Paper Series in Economics
No. 159, December 2022

ISSN 2190-9806

econpapers.wiwi.kit.edu

Maximal Condorcet Domains

A Further Progress Report

Clemens Puppe Arkadii Slinko

December 2022

Abstract

Condorcet domains are sets of preference orders such that the majority relation corresponding to any profile of preferences from the domain is acyclic. The best known examples in economics are the single-peaked, the single-crossing, and the group separable domains. We survey the latest developments in the area since Monjardet's magisterial overview (2009), provide some new results and offer two conjectures concerning unsolved problems. The main goal of the presentation is to illuminate the rich internal structure of the class of maximal Condorcet domains. In an appendix, we present the complete classification of all maximal Condorcet domains on four alternatives obtained by Dittrich (2018).

Contents

1 Introduction

A Condorcet domain is a set of linear orders on a given set of alternatives such that, if every voter is known to have a preference from that set, the pairwise majority relation is acyclic. Equivalently, Condorcet domains guarantee that a Condorcet winner exists on every subset of alternatives. From the perspective of collective decision making, Condorcet domains thus represent the most regular restricted domains and are therefore of great importance in applications. In particular, every Condorcet domain admits rich classes of Arrovian aggregators as well as strategy-proof social choice functions, see e.g., Moulin (1980); Saporiti (2009); Puppe and Slinko (2019).

Well-known examples of Condorcet domains are the single-peaked domain, the single-crossing domain and the domain of group separable preferences. Besides continuing and growing work on these 'classical' domains, there has also been an unflagging interest in identifying 'large' Condorcet domains, see Abello and Johnson (1984); Craven (1996); Fishburn (1997, 2002) and the literature surveyed by Monjardet (2009). Among other things, this literature has demonstrated the great diversity of Condorcet domains, the scope of which is far from being understood completely. However, progress has been made on several counts since Monjardet's magisterial survey was published, and it seems to us that the time is ripe to summarize this progress. This is the purpose of the present paper.

First, the work of Danilov et al. (2012) has led to a deeper understanding of an important subclass of Condorcet domains, the so-called 'peak-pit' domains (with maximal width) of which the single-peaked, the single-crossing and the 'Fishburn alternating scheme domains' are special cases. Moreover, the subclass of Black and Arrow single-peaked domains have been axiomatically characterized by Puppe (2018) and Slinko (2019), respectively, and are now well understood.

Second, an important connection with the theory of median graphs has been established in Puppe and Slinko (2019) that helps understanding the structure of Condorcet domains by visualizing them as graphs.

Third, a complete list of all maximal Condorcet domains on up to five alternatives has been obtained by Dittrich (2018) using a computational protocol: there are exactly 18 different equivalence classes in the case of four alternatives, and 688 in the case of five alternatives. We present the complete classification for the case of four alternatives in the appendix of this paper. By contrast, a detailed qualitative analysis of the class of all maximal domains on five alternatives is clearly beyond the scope and limits of the present survey; here, we only review the classification of all peak-pit domains with maximal width in the case of five alternatives obtained by Li et al. (2021).

Fourth, Danilov and Koshevoy (2013) discovered a series of symmetric maximal Condorcet domains that for any number of alternatives have the size of just 4; we call them *Raynaud domains*, as Raynaud (1981) was the first who discovered such a domain in the case of four alternatives. Raynaud domains were characterized by Karpov and Slinko (2022b) by means of simple permutations, a well-known object in combinatorics. We conjecture that any symmetric maximal domain can be constructed from Raynaud domains and the trivial ones.

Finally, recent progress has also been made in the construction of large peak-pit Condorcet domains by Karpov and Slinko (2022a). The long-standing 'Fishburn hypothesis' was that Fishburn's so-called 'alternating scheme' produces the largest peak-pit domains of maximal width for all numbers of alternatives. But this was refuted by Danilov et al. (2012) who showed that for 42 alternatives the conjecture is false. Karpov and Slinko (2022a) proved that Fishburn's conjecture is false already for 34 alternatives. At the heart of both results is a construction that produces from two given peak-pit maximal Condorcet domains another maximal peak-pit domain of larger size.

Importantly, the present survey is limited to the case of Condorcet domains of *linear* orders, and to the analysis of *maximal* Condorcet domains. A first step in the analysis of Condorcet domains of weak orders is taken in Puppe (2018), and some remarkable facts about 'closed' (but not necessarily maximal) Condorcet domains have been established in Puppe and Slinko (2019).

The remainder of this survey is organized as follows. In Section 2 we review the basic concepts and results on Condorcet domains. Section 3 analyzes the class of connected domains and shows that this class is intimately related to the so-called 'peak-pit' domains. In Subsection 3.1, we present the characterization of the 'Black' single-peaked domain as the only maximal Condorcet domain that satisfies three simple conditions: connectedness, maximal width (i.e., the existence of two completely reversed orders) and minimal richness (i.e., for each alternative the existence of an order with that alternative at the top). Subsection 3.2 contains the generalization of this result which dispenses with the maximal width condition and shows that the remaining two conditions characterize the so-called 'Arrow' single-peaked domains. Subsection 3.3 is devoted to three equivalent combinatorial characterizations of the general 'peak-pit' domains with maximal width: in terms of rhombus tilings (Danilov et al., 2012), in terms of arrangements of pseudolines (Galambos and Reiner, 2008), and in terms of separated ideals (Li et al., 2021). In Subsection 3.4 we present the conjecture that the class of all peak-pit maximal Condorcet domains (with or without maximal width) coincides with the class of connected maximal Condorcet domains. Section 4 is devoted to the class of symmetric Condorcet domains. A particular feature of these domains is that they are characterized by a complete set of 'never-middle' conditions (i.e., the conditions that require that in the restriction to every triple of alternatives there is one alternative that never occupies the middle position in the ranking). We present a construction of indecomposable symmetric Condorcet domains via 'simple sequences' and conjecture that this construction exhausts the class of non-trivial indecomposable symmetric Condorcet domains. Section 5 is devoted to recent advances in the construction of 'large' Condorcet domains, i.e., of maximal Condorcet domains with a large number of different orders.

We do not reproduce proofs that can be found in the literature, but whenever appropriate we do provide the basic intuition behind results.

2 Basic concepts and results

2.1 Isomorphism and flip-isomorphism

Let A be a finite set and $\mathcal{L}(A)$ be the set of all linear orders (transitive, complete and asymmetric binary relations) on A. Any non-empty subset $\mathcal{D} \subseteq \mathcal{L}(A)$ will be called a *domain*. If $a_1 \succ a_2 \succ \cdots \succ a_m$ is a linear order on A, it will be denoted by a string $a_1a_2 \ldots a_m$. We will sometimes denote linear orders also with small latin letters, and write $a \succ_v b$ if a is ranked higher than b in the linear order v (usually associated with a voter). Let us also introduce notation for reversing orders, i.e., if $v = a_1a_2 \ldots a_m$, then $\overline{v} = a_m a_{m-1} \ldots a_1$.

Any sequence $P = (\succ_1, \ldots, \succ_n) = (v_1, \ldots, v_n) \in \mathcal{D}^n$ of linear orders from $\mathcal{D}$ will be called a *profile* over $\mathcal{D}$. Unlike a domain it can contain several identical linear orders. Profiles usually represent a collection of opinions of members of a certain society so $v_1, \ldots, v_n$ are also called *voters* (voters and their linear orders are usually denoted with the same letter).

Definition 1. *The* majority relation $\succeq_P$ *of a profile* $P = (\succ_1, \ldots, \succ_n)$ *is defined as*

$$a \succeq_P b \iff \#\{i \mid a \succ_i b\} \geq \#\{i \mid b \succ_i a\},$$

i.e., $a \succeq_P b$ *means that the number of voters who prefer* a *to* b *is at least as large as the number of voters who prefer* b *to* a*. For an odd* n*, this relation is a* tournament*, i.e., complete and asymmetric binary relation.*

The main object of our investigation are Condorcet domains, defined as follows.

Definition 2. *A domain* $\mathcal{D} \subseteq \mathcal{L}(A)$ *over a set of alternatives* A *is a* Condorcet domain *if the majority relation of any profile* P *over* $\mathcal{D}$ *with an odd number of voters*[1] *is transitive. A Condorcet domain* $\mathcal{D}$ *is* maximal *if for any Condorcet domain* $\mathcal{D}' \subseteq \mathcal{L}(A)$ *the inclusion* $\mathcal{D} \subseteq \mathcal{D}'$ *implies* $\mathcal{D} = \mathcal{D}'$*.*

An equivalent definition of a Condorcet domain requires the majority relation to be *acyclic* for all n, see Monjardet (2009); this is why Fishburn (1997, 2002) refers to Condorcet domains also as *acyclic sets of linear orders*.

As any other abstractly defined object in mathematics, a Condorcet domain can have several 'material' appearances. To relate them together we need a concepts of an isomorphism and flip-isomorphism. Let $\psi\colon A \to A'$ be a bijection between two sets of alternatives. It can be extended to a mapping $\psi\colon \mathcal{L}(A) \to \mathcal{L}(A')$ in two ways: by mapping linear order $u = a_1a_2 \ldots a_m$ onto $\psi(u) = \psi(a_1)\psi(a_2) \ldots \psi(a_m)$[2] or to $\overline{\psi(u)} = \psi(a_m)\psi(a_{m-1}) \ldots \psi(a_1)$.

Definition 3. *Let* A *and* A' *be two sets of alternatives (not necessarily distinct) of equal cardinality. We say that two domains,* $\mathcal{D} \subseteq \mathcal{L}(A)$ *and* $\mathcal{D}' \subseteq \mathcal{L}(A')$ *are* isomorphic *if there is a bijection* $\psi\colon A \to A'$ *such that* $\mathcal{D}' = \{\psi(d) \mid d \in \mathcal{D}\}$*, and* flip-isomorphic *if* $\mathcal{D}' = \{\overline{\psi(d)} \mid d \in \mathcal{D}\}$*.*

[1] further referred to as an 'odd profile'.

[2] We use the same notation for both mappings since there can be no confusion.

In the particular case, when $\overline{\mathcal{D}} = \{\overline{u} \mid u \in \mathcal{D}\}$, we call the domain $\overline{\mathcal{D}}$ as *flipped* $\mathcal{D}$; evidently, the flipped domain $\overline{\mathcal{D}}$ is flip-isomorphic to $\mathcal{D}$, but it is usually not isomorphic to it.

There is only one maximal Condorcet domain on a set $\{a, b\}$ of two alternatives, namely $CD = \{ab, ba\}$. Up to an isomorphism, there are three maximal Condorcet domains on a set of three alternatives $\{a, b, c\}$, and up to an isomorphism or a flip-isomorphism there are only two maximal Condorcet domains on a set of three alternatives. To verify and illustrate this, consider the following two pairs of domains. The two domains shown in Fig. 1 belong to the class of so-called group separable domains introduced by Inada (1964).

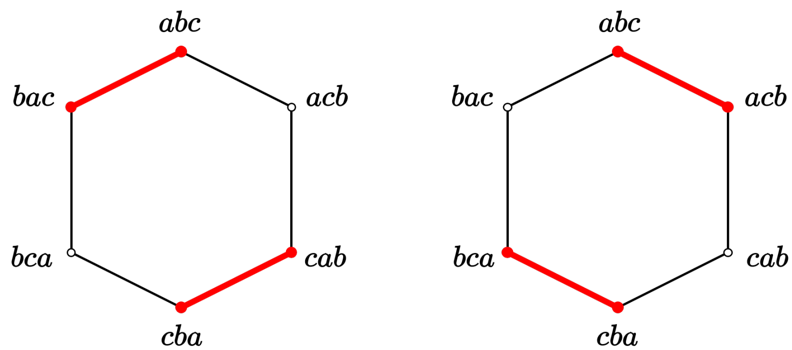


Figure 1: Two isomorphic group separable domains

Definition 4. *A domain $\mathcal{D}$ on the set A of alternatives is called* group separable *if every subset $B \subseteq A$ with at least two elements can be partitioned into two non-empty subsets B' and B'' such that every order in $\mathcal{D}$ either (i) ranks all elements of B' above all elements in B'', or (ii) ranks all elements of B'' above all elements in B'.*

Inada (1964) showed that group separability is a sufficient condition for acyclicity of the majority relation. Indeed, as is easily verified the two domains $\{abc, bac, cab, cba\}$ (Fig. 1, left) and $\{abc, acb, bca, cba\}$ (Fig. 1, right) are maximal Condorcet domains on the set $\{a, b, c\}$. The group separability of the domain on the left side follows via the partition $\{\{a, b\}, \{c\}\}$ of A, and of the domain on the right side via the partition $\{\{a\}, \{b, c\}\}$ (for proper subsets of A the condition is trivial). The first domain is in fact isomorphic to the second, as can be seen by applying the bijection $\psi(a) = c, \psi(b) = b, \psi(c) = a$. Both are self flip-isomorphic relative to the identity mapping.

Fig. 2 depicts two other well-known Condorcet domains on the set $\{a, b, c\}$, the single-peaked and the single-dipped domain.

Definition 5. *A domain $\mathcal{D}$ on the set A of alternatives is called* single-peaked *if there exists a linear order (the 'spectrum') $>$ on A such that, for every order $v \in \mathcal{D}$, the upper contour set $U_v(b) := \{a \in A : a \succ_v b\}$ is connected in the order $>$ for all $b \in A$. The domain of all single-peaked orders with respect to a given spectrum $>$ is denoted by $\mathcal{SP}_>$.*

Similarly, a domain $\mathcal{D}$ on the set A of alternatives is called single-dipped *if there exists a linear order (the 'spectrum') $>$ on A such that, for every order $v \in \mathcal{D}$, the lower contour set $L_v(b) := \{a \in A : b \succ_v a\}$ is connected in the order $>$ for all $b \in A$. The domain of all single-dipped orders with respect to a given spectrum $>$ is denoted by $\mathcal{SD}_>$.*

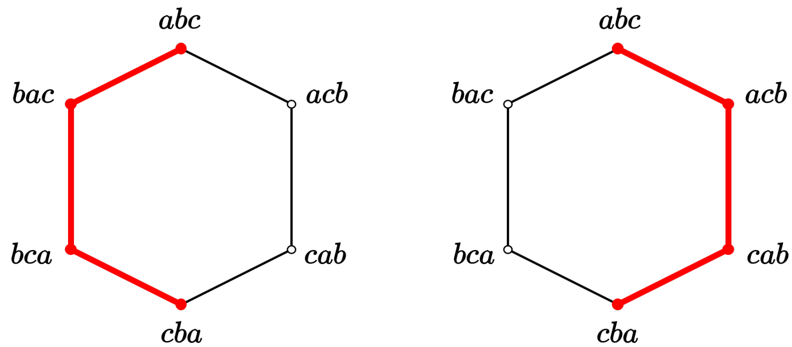


Figure 2: Two flip-isomorphic domains: $\mathcal{SP}_{abc}$ (left) and $\mathcal{SD}_{abc}$ (right)

The two domains shown in Fig. 2 are not isomorphic (this follows, e.g., from the fact that they have a different number of distinct top alternatives), but flip-isomorphic.

Remark 1. *A remark on terminology is in order. The single-peaked domain as defined above has been introduced by Black (1948, 1958) and discussed in Arrow (1951, 1963). A weaker ('local') concept of single-peakedness only requires the restriction of a domain to any triple to be single-peaked, see Sen (1970). To distinguish the two concepts, some authors have called the above 'global' notion of single-peakedness 'Black single-peakedness' and the weaker local version 'Arrow single-peakedness.' We reserve the generic term 'single-peakedness' for the global concept, and distinguish the local version by calling it 'Arrow single-peakedness.' The distinction will play an important role in Section 3 below.*

We will say that a Condorcet domain $\mathcal{D}$ is *closed* if the majority relation corresponding to any odd profile over $\mathcal{D}$ is again an element of $\mathcal{D}$. The following simple observation is useful (see e.g., Puppe and Slinko (2019)).

Proposition 2.1. *Let $\mathcal{D}$ be a Condorcet domain and $v \in \mathcal{L}(A)$ be the majority relation corresponding to an odd profile over $\mathcal{D}$. Then $\mathcal{D} \cup \{v\}$ is again a Condorcet domain. In particular, every Condorcet domain is contained in a closed Condorcet domain and every maximal Condorcet domain is closed.*

2.2 Never Conditions

The domain to the left in Fig. 1 contains all the linear orders on $\{a, b, c\}$ in which c is never ranked second, the domain to the right in Fig. 1 contains all the linear orders on $\{a, b, c\}$ in which a is never ranked second; following Monjardet (2009), we denote these conditions as $cN_{\{a,b,c\}}2$, and $aN_{\{a,b,c\}}2$, respectively. We note that these are the only conditions of type $xN_{\{a,b,c\}}i$ with $x \in \{a,b,c\}$ and $i \in \{1,2,3\}$ that these two domains satisfy. Similarly, the domain to the left in Fig. 2 contains all the linear orders on $\{a,b,c\}$ in which b is never ranked last—that is, it satisfies $bN_{\{a,b,c\}}3$,—and the domain to the right in Fig. 2 contains all the linear orders on $\{a,b,c\}$ in which b is never ranked first satisfying $bN_{\{a,b,c\}}1$. Again, these are the only conditions of type $xN_{\{a,b,c\}}i$ with $x \in \{a,b,c\}$ and $i \in \{1,2,3\}$ that these domains satisfy.

Definition 6. *Any condition of type $xN_{\{a,b,c\}}i$ with $x \in \{a,b,c\}$ and $i \in \{1,2,3\}$ is called a* never condition *since it says that in a triple $\{a,b,c\}$ alternative x never takes i^{th} position. We say that a family $\mathcal{N}$ of*

$$\{xN_{\{a,b,c\}}i \mid \{a,b,c\} \subseteq A,\ x \in \{a,b,c\} \text{ and } i \in \{1,2,3\}\}$$

is a complete set of never-conditions *if $\mathcal{N}$ contains at least one never condition for every triple a,b,c of elements of A.*

Denote by $\mathcal{D}(\mathcal{N})$ the collection of all linear orders that satisfy the set of never conditions $\mathcal{N}$. If $\mathcal{D}(\mathcal{N})$ is non-empty, we call $\mathcal{N}$ *consistent*.[3] Let us also denote by $\mathcal{N}(\mathcal{D})$ the set of all never conditions that are satisfied by all linear orders from $\mathcal{D}$.

Proposition 2.2. *A non-empty domain of linear orders $\mathcal{D} \subseteq \mathcal{L}(A)$ is a Condorcet domain if and only if it satisfies a complete set of never conditions.*[4] *Moreover, for any complete and consistent set of never conditions $\mathcal{N}$ the domain $\mathcal{D}(\mathcal{N})$ is a closed Condorcet domain. Conversely, every maximal Condorcet domain is of this form.*[5]

Remark 2. *Frequently, a maximal Condorcet domain $\mathcal{D}$ satisfies* exactly *one never condition for any given triple $\{a,b,c\}$. In fact, this will be the case whenever the restriction $\mathcal{D}_{\{a,b,c\}}$ of $\mathcal{D}$ to $\{a,b,c\}$ contains four distinct orders. Slinko (2019) calls domains $\mathcal{D}$ with $|\mathcal{D}_{\{a,b,c\}}| = 4$ for all triples $\{a,b,c\}$* copious. *But not all maximal Condorcet domains are copious. Here is an example on five alternatives.*[6] *Let*

$$\mathcal{D} \ := \ \{abcde, caebd, dbeac, edcba\}. \tag{1}$$

The domain $\mathcal{D}$ is a maximal Condorcet domain (see Section 4 below). Its restriction to the triple $\{a,b,d\}$ is given by $\mathcal{D}_{\{a,b,d\}} = \{abd, dba\}$, in particular, $\mathcal{D}$ is not copious; in fact, it satisfies the never conditions $aN_{\{a,b,d\}}2$, $bN_{\{a,b,d\}}1$, $bN_{\{a,b,d\}}3$, and $dN_{\{a,b,d\}}2$.

While not every maximal Condorcet is copious, we have the following result. Say that a Condorcet domain $\mathcal{D}$ is *ample* if, for every pair of distinct alternatives $\{a,b\}$ we have $|\mathcal{D}_{\{a,b\}}| = 2$ which means that some linear orders in $\mathcal{D}$ rank a higher than b while other linear orders rank b higher than a.

[3] It is easy to construct complete sets of never conditions that are inconsistent, i.e., such that $\mathcal{D}(\mathcal{N})$ is empty.

[4] The website https://nevercondition.de offers an online tool to determine, for $m = 4,5,6$ alternatives, if a given domain is a Condorcet domain and, if so, which never conditions it satisfies.

[5] On the other hand, not every closed Condorcet domain must contain *all* orders compatible with a given set of complete and consistent never conditions.

[6] All maximal Condorcet domains on the set of up to four alternatives can be shown to be copious.

Proposition 2.3. *Every maximal Condorcet domain is ample.*

Proof. Suppose $\mathcal{D} \subseteq \mathcal{L}(A)$ is a maximal Condorcet domain that is not ample. Then there exist $a, b \in A$ such that $a \succ_u b$ in every linear order $u \in \mathcal{D}$. Let v be a linear order in which the difference in the ranks of a and b is minimal, $v = \ldots a \ldots b \ldots$. Consider $v' = \ldots ba \ldots$ where v' is obtained from v by moving b just beyond a and identical otherwise. Consider $\mathcal{D}' = \mathcal{D} \cup \{v'\}$.

Let $c \in A$ be arbitrary. Then $\mathcal{D}_{\{a,b,c\}} \subseteq \{abc, acb, cab\}$. Suppose in v we had $c \succ_v a \succ_v b$. Then $\mathcal{D}'_{\{a,b,c\}} \subseteq \{abc, acb, cab, cba\}$ which is a never-top condition. If in v we had $a \succ_v c \succ_v b$, then $\mathcal{D}'_{\{a,b,c\}} \subseteq \{abc, acb, cab, bac\}$ which is never-bottom one. The case if $a \succ_v b \succ_v c$ is similar. Hence, $\mathcal{D}'$ satisfies a complete (and consistent) set of never conditions, i.e., $\mathcal{D}$ was not maximal. $\square$

A domain that, for any triple $\{a, b, c\} \subseteq A$, satisfies a condition $xN_{\{a,b,c\}}1$ with $x \in \{a, b, c\}$ is called *never-top* domain, a domain that for any triple $\{a, b, c\} \subseteq A$ satisfies a condition $xN_{\{a,b,c\}}2$ with $x \in \{a, b, c\}$ is called *never-middle* domain,[7] and a domain that for any triple $\{a, b, c\} \subseteq A$ satisfies a condition $xN_{\{a,b,c\}}3$ with $x \in \{a, b, c\}$ is called *never-bottom* domain. A domain that, for any triple, satisfies either a never-top or a never-bottom condition is called a *peak-pit domain* (Danilov et al., 2012). Both the never-top and the never-bottom conditions are called *peak-pit conditions*.

2.3 Embedding in the Permutohedron

In mathematics, the universal domain $\mathcal{L}(A)$ has many representations. The most useful one for us is by the *permutohedron* of order m, which is an $(m-1)$-dimensional polytope embedded in an m-dimensional space. Its vertices are labeled by the permutations of $\{1, 2, \ldots, m\}$ from the symmetric group S_m. Two permutations are connected by an edge if they differ only in two neighboring places. For our purposes geometry is not important, so for us the permutohedron is the skeleton of this polytope, that is, the graph whose vertices are the permutations from S_m with edges inherited from the edges of the aforementioned polytope. The use of permutohedron in social choice was pioneered by Guilbaud and Rosenstiehl (1963).

The permutahedron is naturally endowed with the following betweenness structure (Kemeny, 1959). An order v is *between* orders u and w if $v \supseteq u \cap w$, i.e., if v agrees with all binary comparisons in which u and w agree (see also (Kemeny and Snell, 1960)). The set of all orders that are between u and w is called the *interval* spanned by u and w and is denoted by $[u, w]$. With this notation, two orders u, w are connected by an edge—we call them neighbors—if and only if $[u, w] = \{u, w\}$.

Figures 1 and 2 show the permutohedron of order three with the different subdomains shown in red (for a graphic representation of the permutohedron of order four, see below).

[7] The domain (1) is a never-middle domain notwithstanding the fact that it *also* satisfies a never-top and a never-bottom condition on the triple $\{a, b, d\}$; indeed, for all other triples, it satisfies exactly one never-middle condition.

One observes an important difference between the two group separable Condorcet domains in Fig. 1 and the single-peaked and single-dipped domains in Fig. 2: the latter are connected but the former are not. We formalize connectedness in the following definition.

Definition 7. *A domain $\mathcal{D}$ is called* connected *if any two orders $u, w \in \mathcal{D}$ are connected by a path in the permutohedron that stays within $\mathcal{D}$, in other words, if there exist $\{v_1, \ldots, v_k\} \subseteq \mathcal{D}$ such that, for all $j = 1, \ldots, k-1$, v_j and v_{j+1} differ only in two neighboring positions, and $v_1 = u$ and $v_k = w$. A stronger notion of* direct connectedness *requires that the path between u and w satisfies $\{v_h, ..., v_l\} \subseteq [v_h, v_l]$ for all $1 \leq h < l \leq k$ (Sato, 2013; Puppe, 2016).*

3 Connected Domains

3.1 The Single-Peaked Domain

The single-peaked domain $\mathcal{SP}_>$ is connected (see e.g., Elkind et al. (2014)) and possesses the following two properties.

Definition 8. *A Condorcet domain $\mathcal{D}$ is said to have* maximal width *if together with some linear order u it also contains $\overline{u}$.*

The property of maximal width plays an important role in the analysis of Condorcet domains as the existence of two completely reversed orders simplifies the matters, sometimes considerably.[8] This is reflected by the fact that a maximal Condorcet domain can be naturally endowed with the structure of a distributive lattice if and only if it has maximal width, see Corollary 3.2 in Puppe and Slinko (2019) and the references there. Also observe that a maximal connected Condorcet domain can contain at most one pair of completely reversed orders. Indeed, if we had two such pairs, say u, $\overline{u}$ and v, $\overline{v}$, then we can find a triple $\{a, b, c\}$ on which u and v disagree. Without loss of generality we may assume that the restriction $u_{\{a,b,c\}}$ of u and the restriction $v_{\{a,b,c\}}$ of v are, respectively, abc and acb. But then $\mathcal{D}_{\{a,b,c\}} = \{abc, cba, acb, bca\}$ which is not connected. In particular, a maximal connected (or peak-pit) Condorcet domain with maximal width contains a *unique* pair of completely reversed orders.

The second property of the single-peaked domain is its 'minimal richness,' as follows.

Definition 9. *A Condorcet domain $\mathcal{D}$ is said to be* minimally rich *if, for all $a \in A$, there exists an order in $\mathcal{D}$ that has alternative a on top.*

From an economic point of view minimal richness is a very natural property, and it has been considered frequently in the literature on domain restrictions.

The following result shows that the single-peaked domain is the only connected maximal Condorcet domain to have these two additional properties. The (median) graph of the single-peaked domain $\mathcal{SP}_{abcd}$ is depicted in Fig. 3, its embedding in the 4-permutohedron is shown in Fig. 4.

[8] Danilov and Koshevoy (2013) refer to domains with maximal width as 'normal.'

Theorem 3.1 (Puppe (2018)). *A maximal Condorcet domain $\mathcal{D}$ is connected, minimally rich and has maximal width if and only if there exists a linear order $>$ on A such that $\mathcal{D} = \mathcal{SP}_>$.*

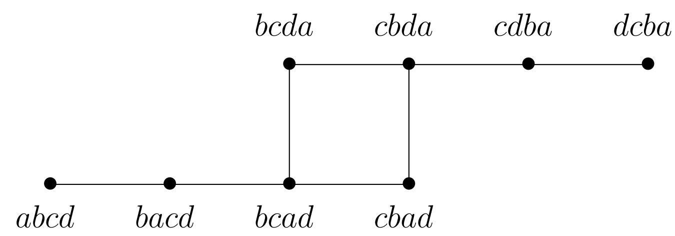


Figure 3: The graph of the single-peaked domain $\mathcal{SP}_{abcd}$.

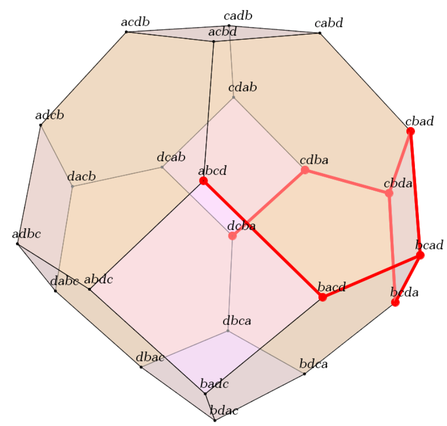


Figure 4: Embedding of the single-peaked domain $\mathcal{SP}_{abcd}$ in the 4-permutohedron

The idea of the proof of Theorem 3.1 is simple. All three properties of connectedness, maximal width and minimal richness are inherited from a domain $\mathcal{D}$ to its restrictions on every triple. But the only domain on a triple that has these properties is the single-peaked domain (see Fig. 2, left). Hence, the restriction of $\mathcal{D}$ to every triple is single-peaked, i.e., satisfies a (unique) never-bottom condition; the maximal width condition guarantees that these never-bottom conditions are satisfied with respect to a common spectrum.

A similar argument implies that there is a unique maximal connected Condorcet domain $\mathcal{D}$ that has maximal width and has the property that every alternative occurs at least once at the bottom of the orders in $\mathcal{D}$, namely the single-dipped domain (cf. Fig. 2, right).

A different characterization of the single-peaked domain in terms of 'sign representations' of single-peaked orders has been given by Zhan (2022). To describe it, consider $A = \{1, 2, , \ldots, n\}$ and the class of all single-peaked orders $\mathcal{SP}_>$ with respect to the natural order $>$ on $\{1, 2, ..., n\}$. Let $v = a_1 \ldots a_n$ be a single-peaked order with peak a_1. If the second ranked alternative a_2 is such that $a_2 > a_1$, then the first sign in the sequence corresponding to v is a $+$; if $a_2 < a_1$, then it is a $-$. Now, suppose that we have already constructed a string of $k-1$ signs from the set $\{+,-\}$ for the suborder $a_1 \ldots a_k$. The next ranked alternative a_{k+1} either satisfies $a_{k+1} > a_j$ for all $j = 1, \ldots, k$, in which case the k^{th} sign in the sequence is $+$, or it satisfies $a_{k+1} < a_j$ for all $j = 1, \ldots, k$, in which case the k^{th} sign is $-$. Continuing this way we obtain a sequence of $+$ and $-$ of length $n-1$ that uniquely encodes v.

On the other hand, if we have a sequence of $+$ and $-$, then counting the number of $+$ signs gives us the top alternative of the corresponding order: if there are k plusses, then the top preference is $n-k$. From the top alternative, the sequence of $+$ and $-$ determines then the entire order in a straightforward manner. For example, with $n = 5$, the single-peaked orders 34251 and 43251 can be encoded as $+-+-$ and $--+-$, respectively. Conversely, $++-+$ denotes the single-peaked preference 23415. Specifically, we have the following result.

Proposition 3.1 (Zhan (2022))**.** *The domain of single-peaked orders $\mathcal{SP}_>$ on a set A of cardinality n is in a bijective correspondence with the set of strings of signs $+$ and $-$ of length $n-1$. In particular, we have $|\mathcal{SP}_>| = 2^{n-1}$.*

3.2 The Arrow Single-Peaked Domains

While the property of minimal richness has a clear economic meaning, the maximal width condition is arguably less attractive in applications. It is thus natural to ask what happens if we drop the maximal width condition in the characterization result stated in the previous subsection.

Call a non-empty domain $\mathcal{D}$ an *Arrow single-peaked* domain if $\mathcal{D}$ satisfies a complete set of never-bottom conditions. It has long been known that the condition of single-peakedness on all triples does by itself not imply single-peakedness with respect to a common spectrum (cf. Sen (1970)).[9] The following result shows that the Arrow single-peaked domains are exactly the maximal connected Condorcet domains that are minimally rich. Fig. 5 shows the graph of an Arrow single-peaked maximal Condorcet domain without maximal width.

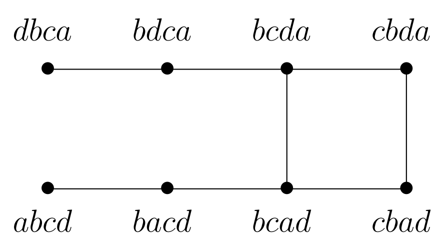


Figure 5: Graph of an Arrow single-peaked domain without maximal width

[9] The characterization given by Ballester and Haeringer (2011) identifies the additional property on quadruples needed for this implication.

Theorem 3.2 (Slinko (2019)). *A maximal Condorcet domain is connected and minimally rich if and only if it is Arrow single-peaked.*

In Slinko (2019) it is also shown that all maximal Arrow single-peaked domain are copious and have cardinality $2^{|A|-1}$. Unlike the classical Black maximal single-peaked domains which are unique up to isomorphism or flip-isomorphism for each number of alternatives, the number of Arrow single-peaked domains grows rapidly. In an unpublished paper, Leversidge (2019) showed that the number ASP(n) of Arrow single-peaked Condorcet domains for $m \in \{3, \ldots, 8\}$ is given by the following table:

n	3	4	5	6	7	8
ASP(n)	1	2	6	40	560	17024

Table 1: The number of maximal Arrow single-peaked domains depending on the number of alternatives.

3.3 Connected Domains with Maximal Width

In this subsection, we study the class of connected maximal Condorcet domains that are not necessarily minimally rich but still satisfy the maximal width conditions. For this it will often be convenient to identify A with the set $\{1, 2, \ldots, m\}$. The two most prominent examples of such domains, besides the single-peaked domain, are the single-crossing domains and the so-called Fishburn's alternating scheme domains.

3.3.1 Maximal Condorcet Domains that are Single-Crossing

A frequently useful sufficient condition for transitivity of the majority relation is the 'single-crossing property'; it requires that the orders in a domain can be put in a sequence so that along this sequence the relative positions of any pair of alternatives is reversed at most once. Roberts (1977) and Gans and Smart (1996) provide a number of economic applications of this property. Single-crossing domains have a number of attractive characteristics. For instance, in every group with an odd number of voters with preferences from a single-crossing domain there is always a voter whose preference coincides with the majority relation — this fact is known as the 'Representative Voter Theorem' (Grandmont, 1978; Rothstein, 1991). Moreover, the collective choice prescribed by the majority relation can be implemented in dominant strategies through a simple mechanism (Tohmé and Saporiti, 2006), among the many social choice functions implementable in dominant strategies on single-crossing domains (Saporiti, 2009).

Recent research has revealed that understanding single-crossing domains could be crucial to understanding Condorcet domains in general. Indeed, Galambos and Reiner (2008) proved that any connected maximal Condorcet domain of maximal width is a *union* of single-crossing domains. An important question then is: under which conditions is a single-crossing domain by itself already a maximal Condorcet domain? We will provide two (related) answers to this question in this subsection.

Let us start with the formal definition of single-crossingness.

Definition 10. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is said to be a* single-crossing domain *if the orders from $\mathcal{D}$ can be written in a sequence $(\succ_1, \dots, \succ_{|\mathcal{D}|})$ so that $i \succ_1 j$ implies either $i \succ_s j$ for every s, or there is an integer k such that $i \succ_s j$ for every $s \leq k$ and $j \succ_s i$ for every $s > k$. Simply put, traveling along $\succ_1$, $\succ_2, \dots$ the relative positions of i and j swap at most once. If $\mathcal{D}$ is not a proper subset of another single-crossing domain, then we say it is a* maximal single-crossing domain.[10]

As an example, let us consider the domain $\mathcal{D}$ on $A = \{1, 2, 3, 4\}$ whose orders are represented as columns of the following matrix

$$\begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{2}$$

Observe that each order can be obtained from its immediate predecessor by swapping exactly one pair of neighboring alternatives: the second order is obtained from the first by swapping the pair $(1, 2)$, the third from the second by swapping $(1, 3)$, the fourth from the third by swapping $(1, 4)$, the fifth from the fourth by swapping $(3, 4)$, the sixth from the fifth by swapping $(2, 4)$, and finally the seventh from the sixth by swapping $(2, 3)$. Consequently, the graph corresponding to this domain is a line graph.

The following result summarizes the basic properties of single-crossing domains. Say that a domain $\mathcal{D}$ has the *representative voter property* if, for all profiles $(\succ_1, \dots, \succ_n)$ with odd n, there exists $k \in \{1, \dots, n\}$ such that the pairwise majority relation corresponding to the profile coincides with $\succ_k$.

Proposition 3.2. **a)** *Every single-crossing domain has the representative voter property. In particular, every single-crossing domain is a Condorcet domain.*

b) *Every maximal single-crossing domain is connected and has maximal width.*

c) *A domain on a set of m alternatives is a maximal single-crossing domain if and only if its associated graph is a line graph of length $\frac{1}{2}m(m-1) + 1$.*

d) *Every single-crossing domain on at least four alternatives that is a maximal Condorcet domain is a 'proper' peak-pit domain (i.e., it is copious and satisfies some never-bottom as well as some never-top conditions but no never-middle ones).*

Parts a) - c) of Proposition 3.2 follow from the analysis in Puppe and Slinko (2019).[11] Part d) follows from two observations. First, every connected domain of maximal width is a peak-pit domain and copious. Second, if only never-top or only never-bottom conditions

[10] Observe that a maximal single-crossing domain need not be maximal as a Condorcet domain.

[11] Theorem 6 in Puppe and Slinko (2019) shows that a maximal Condorcet domain has the representative voter property if and only if it is either single-crossing or one of the 4-point domains discussed in Section 4 below.

were satisfied, the domain would be either single-peaked or single-dipped; however, it can be neither of these (this follows at once from a comparison of their sizes). Moreover, Slinko et al. (2021) characterize exactly the set of never conditions that a single-crossing maximal Condorcet domain satisfies.

Every maximal single-crossing domain on a set of m alternatives has $k = \frac{1}{2}m(m-1)+1$ elements[12] and is characterized by a sequence of $k-1$ *pairs of swapped alternatives*

$$(i_1, j_1),\ (i_2, j_2),\ \ldots, (i_{k-1}, j_{k-1}) \tag{3}$$

from the set $\{(i,j) \mid 1 \le i < j \le n\}$. The pair (i_s, j_s) in this sequence means that i_s and j_s are neighbors in $\succ_s$ and $\succ_{s+1}$, with $i_s \succ_t j_s$ for $t = 1, \ldots, s$, and $j_s \succ_t i_s$ for $t = s+1, \ldots, k$, while all other relations between alternatives in $\succ_s$ and $\succ_{s+1}$ are identical. Roughly speaking, the passage from $\succ_s$ to $\succ_{s+1}$ is a swap of neighbors i_s and j_s. For instance, as already noted above, the swapping sequence for the domain (2) is $(1,4)$, $(1,3)$, $(1,4)$, $(3,4)$, $(2,4)$, $(2,3)$.

The following result can be inferred from (Galambos and Reiner, 2008, Th. 2) and is explicitly stated as Theorem 9 in Puppe and Slinko (2019).

Theorem 3.3. *A maximal single-crossing domain $\mathcal{D}$ is a maximal Condorcet domain if and only if the swapping sequence* (3) *characterizing $\mathcal{D}$ satisfies the following 'pairwise concatenation' property:*[13]

$$\{i_s, j_s\} \cap \{i_{s+1}, j_{s+1}\} \neq \emptyset \textit{ for every } s \in \{1, 2, \ldots, k-1\}. \tag{4}$$

The pairwise concatenation property (4) imposes a very rigid structure on a Condorcet domain that can alternatively be described via the notion of a *relay* introduced in Slinko et al. (2021). Let us use an example to illustrate what a relay looks like. In this example $A = \{1, 2, \ldots, 7\}$ and the domain is represented by the following matrix where each column corresponds to an order:

$$\begin{bmatrix}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
2 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 7 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & 6 & 6 & 6 \\
3 & 3 & 1 & 4 & 4 & 4 & 4 & 4 & 4 & 7 & 3 & 3 & 2 & 4 & 4 & 4 & 4 & 6 & 3 & 4 & 4 & 5 \\
4 & 4 & 4 & 1 & 5 & 5 & 5 & 5 & 7 & 4 & 4 & 4 & 4 & 2 & 5 & 5 & 6 & 4 & 4 & 3 & 5 & 4 \\
5 & 5 & 5 & 5 & 1 & 6 & 6 & 7 & 5 & 5 & 5 & 5 & 5 & 5 & 2 & 6 & 5 & 5 & 5 & 5 & 3 & 3 \\
6 & 6 & 6 & 6 & 6 & 1 & 7 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
7 & 7 & 7 & 7 & 7 & 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$

This domain is a maximal single-crossing domain as it has $\frac{7 \cdot 6}{2} + 1 = 22$ orders. It evidently satisfies the pairwise concatenation condition, and is therefore maximal as a Condorcet domain by Theorem 3.3.

In addition, with the help of the red-coloring, it is not difficult to see that the left-to-right procession of preferences follows a distinct pattern that leaves behind an undulating

[12] This is the number of pairs to be switched plus one.

[13] This condition appeared already in the unpublished lecture notes Monjardet (2007).

trajectory like a damping wave. In particular, focusing on the red-colored alternatives, we see that the procession starts with the movement of alternative 1 that keeps going down from the top until it reaches the bottom. Then alternative 7, which occupies the bottom just before, as if having received a relay baton from alternative 1 when they meet, starts moving up until it reaches the top. As alternative 7 reaches the top, the then top alternative, 2, starts to move down. However, instead of stopping at the bottom, alternative 2 stops at second-to-bottom position, handing the baton to the then second-to-bottom alternative, 6, which starts to go up until reaching second-to-top position. This to and fro relay run continues, each leg ending with the initial kth-to-top alternative reaching the kth-to-bottom position, or the kth-to-bottom alternative reaching the kth-to-top position, until, eventually, the initial ranking is completely reversed. The red trajectory is undulating because of the to and fro relay motion, and it is damping because a later runner covers a shorter distance than an earlier runner.

The following characterization is due to Slinko et al. (2021); for the precise mathematical definition of a 'relay representation' we refer to that paper.

Theorem 3.4 (Slinko et al. (2021))**.** *A domain $\mathcal{D}$ is single-crossing and maximal Condorcet if and only if it has a relay representation.*

Summarizing, most maximal single-crossing domains are not maximal as Condorcet domains; in fact, as shown in Slinko et al. (2021), up to isomorphism or flip-isomorphism there is a *unique* single-crossing domain that is at the same time maximal as Condorcet domain. Evidently, among all connected maximal Condorcet domains with maximal width the single-crossing is the one with the minimal number of elements (namely $|A|(|A|-1)/2$). Note finally, that the relay representation shows that a single-crossing domain that is maximal as a Condorcet domain is necessarily far from being minimally rich.

3.3.2 Fishburn's Alternating Scheme

In search for 'large' Condorcet domains, Fishburn (1997) came up with the following structure of a complete set of never-conditions.[14]

Definition 11. *Let $A = \{1, 2, \ldots, m\}$. A complete set of never-conditions is said to satisfy the* alternating scheme*, if for all $1 \leq i < j < k \leq m$ either*

(i) $jN_{\{i,j,k\}}1$, if j is even, and $jN_{\{i,j,k\}}3$, if j is odd, or

(ii) $jN_{\{i,j,k\}}3$, if j is even, and $jN_{\{i,j,k\}}1$, if j is odd.

The corresponding domains are maximal Condorcet domains which we denote by F_m in case (i) and $\overline{F}_m$ in case (ii). The second domain is flip-isomorphic to the first. In particular, $F_2 = \{12, 21\}$, $F_3 = \{123, 132, 312, 321\}$ and

$$F_4 = \{1234, 1324, 3124, 1342, 3142, 3412, 4312, 3421, 4321\}.$$

[14] According to some sources, the idea stemmed from a private communication with Bernard Monjardet.

The latter has the following graph associated with it:

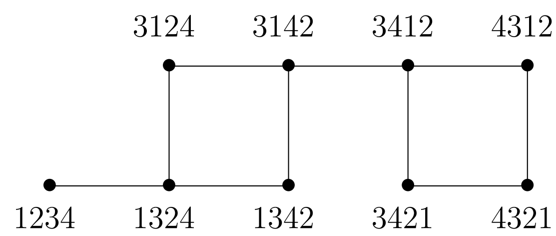


Figure 6: Graph of Fishburn's domain F_4 on four alternatives

Here is an embedding of $\overline{F}_4$ into the permutohedron:

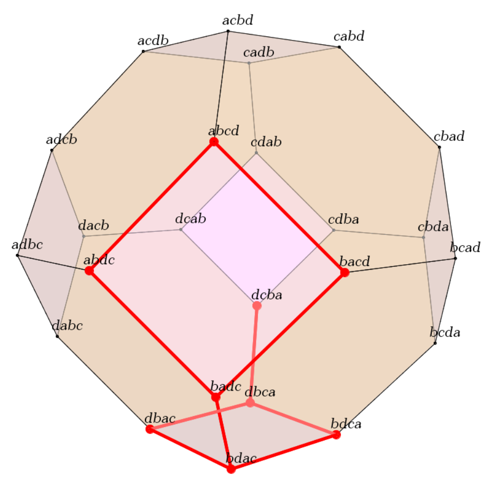


Figure 7: Embedding of $\overline{F}_4$ into the permutohedron

It is easily seen that F_m and $\overline{F}_m$ are connected and have maximal width. Also observe that F_4 has cardinality 9 and is in fact the uniquely largest Condorcet domain on a set of four alternatives up to isomorphism or flip-isomorphism (Raynaud, 1982). Further, it is known that F_m has the uniquely largest cardinality among all maximal Condorcet domains for all $m \leq 7$ (Monjardet, 2009; Galambos and Reiner, 2008). Remarkably, despite the fact that F_m is always strictly larger than the single-peaked domain on m alternatives, the Fishburn domains are never minimally rich as some alternatives are required not to be first in some triples.

3.3.3 Condorcet Domains of the Tiling Type

In order to characterize the class of all connected maximal Condorcet domains with maximal width, Danilov et al. (2012) introduced the 'rhombus tiling' representation of a domain.

Definition 12 (Danilov et al. (2012))**.** *A* rhombus tiling *(or simply a* tiling*) is a subdivision into rhombic tiles of a regular $2m$-gon formed by the points $\sum_i a_i\psi_i$, where $0 \leq a_i \leq 1$ and $\psi_1, \ldots, \psi_m$ are unit vectors in the upper half-plane. This centre-symmetric $2m$-gon has its bottom vertex b at the origin and the top vertex $t = \psi_1 + \ldots + \psi_m$. An ij-tile is a rhombus congruent to the one formed by the points $\lambda\psi_i + \mu\psi_j$, where $0 \leq \lambda, \mu \leq 1$. A* snake *is a path from t to b along the boundaries of the tiles which for each $i = 1, \ldots, m$ contains a unique segment parallel to ψ_i. Each snake corresponds to a linear order on $\{1, \ldots, m\}$ in the following way. If a point traveling from t to b along a snake passes segments parallel to $\psi_{i_1}, \psi_{i_2} \ldots, \psi_{i_m}$, then the corresponding linear order will be $i_1 i_2 \ldots i_m$. The set of snakes of a rhombus tiling, thus, defines a domain which is called a* tiling domain*.*

For $m = 3$ we have a hexagon which can be split into rhombus tiles in two different ways as shown in Figure 8.

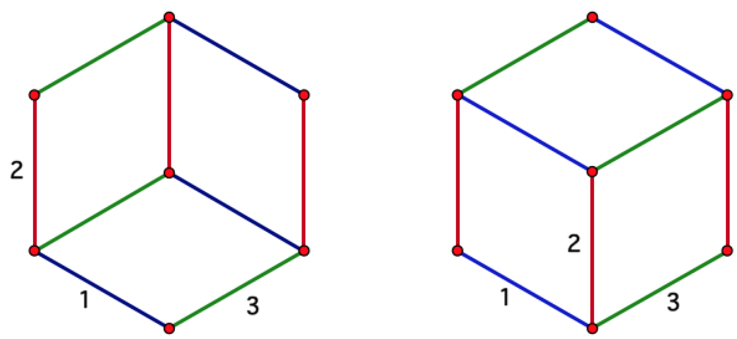


Figure 8: Two tiling domains

These lead to the domains:

$$\{123, 213, 231, 321\} \quad \text{and} \quad \{123, 132, 312, 321\},$$

which are the familiar single-peakedand single-dipped domains, respectively.

Definition 13 (Danilov et al. (2012))**.** *A domain $\mathcal{D}$ containing the two completely reversed orders u and $\overline{u}$ is called* semi-connected *if it contains an entire shortest path in the permutohedron connecting u and $\overline{u}$.*

The following result can be derived from the analysis in Danilov et al. (2012), which, in turn, is based on combinatorial results of Leclerc and Zelevinsky (1998).

Theorem 3.5 (Danilov et al. (2012))**.** *Let $\mathcal{D}$ be a maximal Condorcet domain with maximal width. The following statements are equivalent.*

a) *$\mathcal{D}$ is semi-connected.*

b) *$\mathcal{D}$ is connected.*

c) *$\mathcal{D}$ is a peak-pit domain.*

d) *$\mathcal{D}$ is a tiling domain.*

By Theorem 3.5, the connected maximal Condorcet domains with maximal width are exactly the peak-pit domains with maximal width. Observe, that these domains are all copious and thus satisfy a unique complete set of never conditions. It has been observed in Puppe (2016) that these domains are not only connected but even directly connected, i.e., any two orders of a domain are connected by a *shortest* path in the permutohedron that stays within the domain.

3.3.4 Arrangements of Pseudolines

We now describe an equivalent geometric representation of connected maximal Condorcet domains with maximal width in terms of *pseudoline arrangements* on the plane. This representation has by now become folklore in low dimensional topology, the study of the Yang-Baxter equation and geometric combinatorics (Humphreys, 1994). Galambos and Reiner (2008) were the first to relate these concepts to Condorcet domains.

The most intuitive way to think about an arrangements of pseudolines is geometrically. On two vertical parallel lines L and R in $\mathbb{R}^2$, we mark a set of m equidistant points. The points on the left line are labeled $1, \ldots, m$ in downward order and on the right line the points are marked also $1 \ldots, m$ but in upward order. The two points with the same label i — one on the left and one on the right — are joined by a continuous curve which is called *pseudoline i* so that any two pseudolines intersect exactly at one point, called a *vertex*. The arrangement is *simple* if there is no vertex where three or more pseudolines meet.

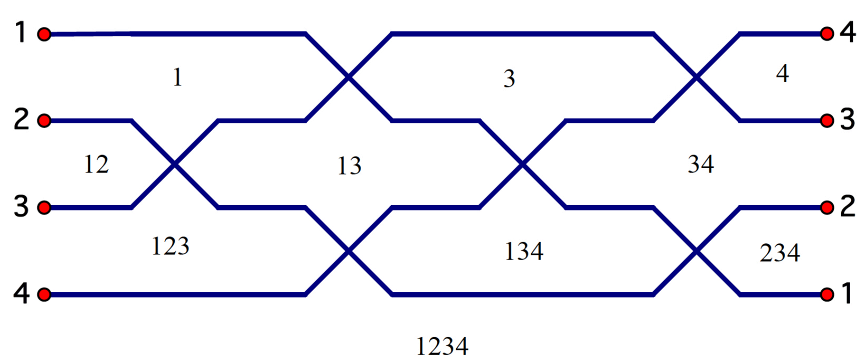


Figure 9: The wiring diagram corresponding to F_4

An arrangement of pseudolines consisting of piecewise linear 'wires' is also called a *wiring diagram*. The wires (pseudolines) are horizontal except for small neighborhoods of their crossings with other wires; see Fig. 9 for an example. There is no loss of generality in assuming that our pseudolines are wires. The arrangements we consider are all simple, and often called *simple numbered arrangements of pseudolines* (Björner et al., 1999, Sect. 6.4).

The parallel lines L and R bound an infinite LR-strip between them. The complement of the pseudolines in the LR-strip is split into *chambers* which are the connected parts of this complement (two of them the top and the bottom ones are unbounded). They are labeled as follows. For a chamber C and any pseudoline k we can say if this chamber is above or below the line k. The label of the chamber is the set of numbers of the pseudolines that go above this chamber (see Fig. 9 for an illustration). By convention, the label $\emptyset$ is attached to the chamber that is above all pseudolines. Every path that connects the upper chamber labeled $\emptyset$ with the bottom chamber labeled $\{1, \ldots, m\}$ and consequently crosses each pseudoline exactly once, naturally defines an order on $\{1, \ldots, m\}$. If it crosses the pseudolines in order $i_1, i_2, \ldots, i_m$, then we attach order $i_1 i_2 \ldots i_m$ to it. The set of all such paths thus defines a domain corresponding to the given simple numbered arrangement of pseudolines. We say that this domain is *represented* by just described arrangement of pseudolines. For instance, as is easily verified, the domain represented by the wiring diagram in Figure 9 is Fishburn's alternating scheme domain F_4.

The representations of connected maximal Condorcet domains of maximal width in terms of rhombus tilings and simple numbered arrangements of pseudolines are 'dual' to each other. This can be inferred from the canonical bijection between the chambers of an arrangement of pseudolines and the vertices of the corresponding rhombus tiling, as indicated in Figure 10 for the case of F_4 (see Elnitsky (1997); Felsner (2012) for more details).

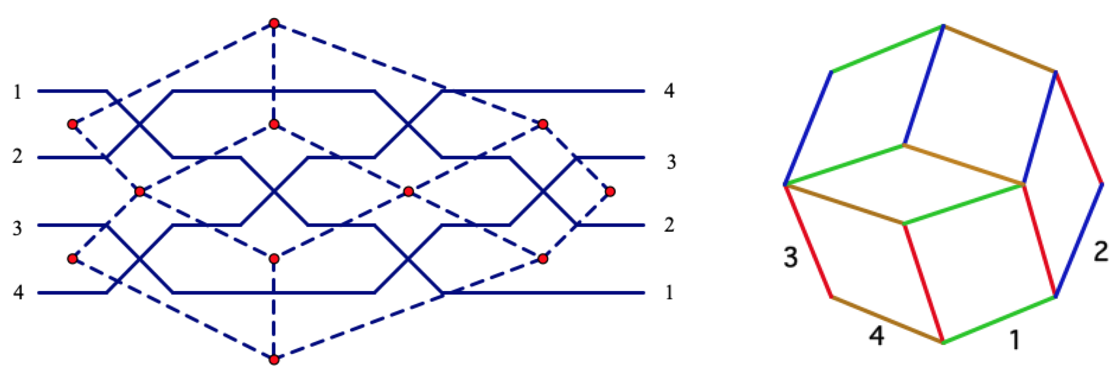


Figure 10: The arrangement of pseudolines and its dual tiling for F_4

Hence we obtain from Theorem 3.5 the following corollary:

Corollary 3.1. *A domain $\mathcal{D}$ is a semi-connected maximal Condorcet domain with maximal width if and only if it can be represented by a simple numbered arrangement of pseudolines.*

3.3.5 Separated Ideals

Finally, we present yet another equivalent way to characterize the connected maximal Condorcet domains with maximal width. The key to this is the observation that the chambers of a simple numbered arrangement of pseudolines correspond to the *initial segments* of the orders in the represented Condorcet domain. Formally, we introduce the notion of an ideal of a domain, as follows. For any order $u = a_1 a_2 \dots a_m \in \mathcal{L}(A)$, denote by $u_k = a_1 a_2 \dots a_k$ the initial segment of length $k \leq m$, and set $\mathrm{Id}_k(u) = \{a_1, \dots, a_k\}$ with $\mathrm{Id}_0(u) = \emptyset$ by convention.

Definition 14. *The* ideal *$Id(\mathcal{D})$ of a domain $\mathcal{D}$ is defined as the collection of all subsets of A that are obtained from initial segments of the orders in $\mathcal{D}$,*

$$Id(\mathcal{D}) = \bigcup_{k=0}^{n} Id_k(\mathcal{D}),$$

where $Id_k(\mathcal{D}) = \{Id_k(u) \mid u \in \mathcal{D}\}$.

Proposition 3.3. *For any connected maximal Condorcet domain $\mathcal{D}$ with maximal width, the ideal $Id(\mathcal{D})$ is given by the family of chambers of the corresponding simple numbered arrangement of pseudolines.*

For instance, for the Fishburn alternating scheme domain F_4 we obtain the ideal

$$\mathrm{Id}(F_4) = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\},$$

which consists of the labels of chambers of the arrangement of pseudolines shown in Fig. 9.

The ideals of connected maximal Condorcet domains with maximal width have a particular structure; indeed, they form a family of 'separated sets' (Leclerc and Zelevinsky, 1998). Specifically, it is said that two sets $X, Y \subseteq A = \{1, 2, \dots, m\}$ are *separated* if there does not exist a triple $\{a, b, c\} \subseteq A$ such that $a < b < c$ and either

$$\begin{array}{lll} (X \cap \{a,b,c\} = \{b\} & \text{and} & Y \cap \{a,b,c\} = \{a,c\}), \text{ or} \\ (X \cap \{a,b,c\} = \{a,c\} & \text{and} & Y \cap \{a,b,c\} = \{b\}). \end{array}$$

A family of subsets is called *separated* if any two sets in the family are separated. The following characterization follows from combining results of Leclerc and Zelevinsky (1998); Danilov et al. (2012); Li et al. (2021).

Theorem 3.6. *Let $\mathcal{D}$ be a maximal Condorcet domain with maximal width. The following statements are equivalent:*

a) *$\mathcal{D}$ is connected;*

b) *$\mathcal{D}$ is a peak-pit domain;*

c) *$Id(\mathcal{D})$ is separated.*

A remarkable corollary of Theorem 3.6 and the correspondence between the ideal of a connected maximal Condorcet domain with maximal width and the chambers of its pseudoline arrangement is the following.

Corollary 3.2 (Li et al. (2021))**.** *Let $\mathcal{D}$ be a maximal Condorcet domain with maximal width on the set of m alternatives. Then,*

$$|Id(\mathcal{D})| \;=\; \frac{m(m+1)}{2}+1.$$

Thus, while the cardinalities of maximal connected Condorcet domains with maximal width on a given number of alternatives can be quite different, the cardinality of the associated ideal is constant. In Li et al. (2021) this fact is explored in order to provide a complete classification of all connected maximal Condorcet domains with maximal width on five alternatives. It turns out that, up to an isomorphism and flip-isomorphism, there are exactly 18 of them with cardinalities ranging from 11 (for the single-crossing) to 20 (for the Fishburn alternating scheme domain F_5); by Corollary 3.2, their ideals all have the same cardinality of $(5 \cdot 6)/2 + 1 = 16$.

3.4 Connected Domains without Maximal Width: A Conjecture

The characterization results provided in the previous section for the connected maximal Condorcet domains all rely crucially on the maximal width assumption. As noted above, this assumption implies, in particular, that the domains have the structure of a distributive lattice; more importantly, it implies that the domains are copious and hence satisfy a unique complete set of never conditions. As shown, these must consist of peak-pit conditions. Even without the maximal width condition, we still have:

Proposition 3.4. *Every connected maximal Condorcet domain satisfies a complete set of peak-pit conditions.*

Proof. As is easily verified, the restriction $\mathcal{D}_{\{x,y,z\}}$ of a connected domain $\mathcal{D}$ to every triple $\{x, y, z\}$ is also connected. Moreover, to avoid cycles, this restriction can contain at most four orders. Then we can check that every connected subdomain on three alternatives with no more than four elements satisfies at least one never-bottom or one never-top condition. □

By Theorem 3.2 the converse statement holds for all maximal Condorcet domains that satisfy either a complete set of never-bottom conditions (the Arrow single-peaked domains) or, by symmetric arguments, a complete set of never-top conditions (the Arrow single-dipped domains): these domains are connected no matter if we have maximal width or not. Together with Proposition 3.4, this observation naturally leads to the following conjecture (which can indeed be verified to hold for all domains on up to 5 alternatives).

Conjecture 1. *A maximal Condorcet domain is connected if and only if it is a peak-pit domain.*

For four alternatives, a classification of all peak-pit domains has been carried out in Dittrich (2018) by computerized search. There, we encounter the following two previously unseen creatures, which (in accordance with our conjecture) are connected.

1. The **ladder domain** is defined by the following complete set of never-conditions:

$$3N_{\{1,2,3\}}1, \quad 4N_{\{1,2,4\}}1, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

It is copious but does not have maximal width (cf. domain $\mathcal{D}_5$ in the appendix).

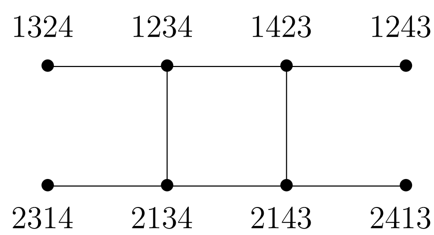


Figure 11: Graph of the ladder domain.

2. The **broken ladder domain** is defined by the following complete set of never-conditions:

$$3N_{\{1,2,3\}}1, \quad 1N_{\{1,2,4\}}3, \quad 1N_{\{1,3,4\}}3, \quad 2N_{\{2,3,4\}}3.$$

It is copious but does not have maximal width (cf. domain $\mathcal{D}_6$ in the appendix).

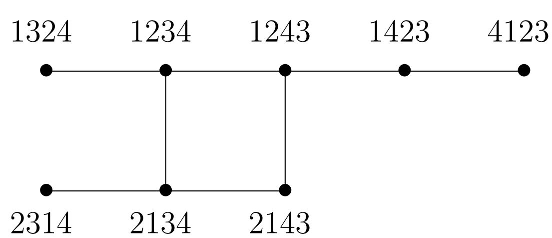


Figure 12: Graph of the Broken Ladder domain.

Proposition 3.5 (Dittrich (2018))**.** *If $m = 4$, then any maximal connected Condorcet domain is either isomorphic or flip-isomorphic to one of the following: the Black single-peaked domain; the single-crossing domain; the Fishburn's domain; the Arrow single-peaked domain without maximal width; the ladder domain; the broken ladder domain. Only the first three of these have maximal width.*

One difficulty in trying to prove Conjecture in general is that maximal Condorcet domains on more than four alternatives need not be copious. Indeed, the phenomenon that maximal Condorcet domains may satisfy different never conditions on the same triple of alternatives (and thus induce less than four different restricted orders on that triple) is not limited to 'small' domains such as the one in (1) above. Here is a non-copious maximal Condorcet domain on five alternatives with cardinality 15 (in the list of all 688 equivalence classes of non isomorphic or flip-isomorphic maximal Condorcet domains on five alternatives obtained by Dittrich (2018) it appears as no. 273).

$$\begin{aligned}\mathcal{D}_{\#273} \;=\; \{&abcde, acbde, acdbe, aebcd, aecbd, aecdb, eabcd, eacbd,\\ &eacdb, ecabd, ecadb, ecdab, ecdba, edcab, edcba\}.\end{aligned}$$

This domain satisfies the multiple never conditions $dN_{\{c,d,e\}}1$ and $eN_{\{c,d,e\}}2$ on the triple $\{c, d, e\}$. Note that although the domain $\mathcal{D}_{\#273}$ thus satisfies a never-top condition on the triple $\{c, d, e\}$, its restriction to this triple is not connected because it *also* satisfies a particular never-middle condition on that triple. In particular, the domain $\mathcal{D}_{\#273}$ is itself not connected. But it does not represent a counterexample to the conjecture because it also satisfies the unique never-middle conditions $eN_{\{b,c,e\}}2$ and $eN_{\{b,d,e\}}2$, hence it is not a peak-pit domain.

The domain $\mathcal{D}_{\#273}$ has maximal width; here is a non-copious maximal Condorcet domain on five alternatives without maximal width. It has cardinality 16, and it is the uniquely largest domain on five alternatives that is not copious (No. 332 in the above mentioned list).

$$\begin{aligned}\mathcal{D}_{\#332} \;=\; \{&abcde, abced, abecd, abedc, acebd, acedb, aecbd, aecdb,\\ &bacde, baced, baecd, baedc, caebd, caedb, eacbd, eacdb\}.\end{aligned}$$

This domain satisfies multiple never conditions $aN_{\{a,b,d\}}3$ and $dN_{\{a,b,d\}}1$ on triple $\{a, b, d\}$, multiple never conditions $aN_{\{a,c,d\}}3$ and $dN_{\{a,c,d\}}1$ on triple $\{a, c, d\}$ and multiple never conditions $aN_{\{a,d,e\}}3$ and $dN_{\{a,d,e\}}1$ on triple $\{a, c, d\}$; it is not connected because it satisfies a unique never middle condition $bN_{\{b,c,e\}}2$ on triple $\{b, c, e\}$.

4 Symmetric Maximal Condorcet Domains

We now turn to the remaining class of 'pure' Condorcet domains, the maximal Condorcet domains that satisfy a complete set of never-middle conditions. These domains turn out to be intimately related to the symmetric domains systematically studied by Danilov and Koshevoy (2013).

Definition 15. *A domain $\mathcal{D}$ is* symmetric *if $u \in \mathcal{D}$ implies $\overline{u} \in \mathcal{D}$.*

Proposition 4.1. *Every symmetric Condorcet domain $\mathcal{D}$ satisfies a complete set of never-middle conditions. Conversely, every never-middle maximal Condorcet domain is symmetric.*

Proof. Suppose that $\mathcal{D}$ is symmetric and satisfies $aN_{\{a,b,c\}}3$ for some triple $a, b, c \in A$. Then $\mathcal{D}_{\{a,b,c\}} \subseteq \{abc, bac, acb, cab\}$. Since $\overline{abc}$ and $\overline{acb}$ are not in $\mathcal{D}_{\{a,b,c\}}$, we must in fact have $\mathcal{D}_{\{a,b,c\}} \subseteq \{bac, cab\}$ by symmetry. In that case, $\mathcal{D}$ also satisfies $bN_{\{a,b,c\}}2$. A similar argument holds if $\mathcal{D}$ satisfies $aN_{\{a,b,c\}}1$.

Conversely, suppose $\mathcal{D}$ satisfies a complete set of never-middle conditions. Since any never-middle condition is itself symmetric, if u satisfies a given never-middle condition so does $\overline{u}$. Due to the maximality of $\mathcal{D}$, if $u \in \mathcal{D}$ then also $\overline{u} \in \mathcal{D}$. □

4.1 Decomposable Domains

Symmetric maximal Condorcet domains are frequently 'decomposable' in the following sense (Karpov and Slinko, 2022b).

Definition 16. *Let $\mathcal{E}$ be a Condorcet domain on the m-element set of alternatives $B = \{b_1, \ldots, b_m\}$. Let also $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets $C_1, \ldots, C_m$ of alternatives. Then we define the domain on $C_1 \cup \ldots \cup C_m$ as*

$$\mathcal{E}(b_1 \to \mathcal{D}_1, \ldots, b_m \to \mathcal{D}_m) \ := \ \{u_1 \ldots u_m \mid u_j \in \mathcal{D}_{i_j} \text{ and } \ b_{i_1} \ldots b_{i_m} \in \mathcal{E}\}.$$

When it can cause no confusion, we will denote this domain as $\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$. We call $\mathcal{E}$ the top-level domain *and $\mathcal{D}_1, \ldots, \mathcal{D}_m$* ground level domains*.*

This definition is similar, in spirit, to the definition of the wreath product of permutations introduced in Atkinson and Stitt (2002).

Definition 17. *A domain $\mathcal{D} \subseteq \mathcal{L}(A)$ is called* decomposable *if it is isomorphic to $\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$, where $|C_i| > 1$ for at least one $\mathcal{D}_i$ where $i \in \{1, \ldots, m\}$.*

Proposition 4.2 (Karpov and Slinko (2022b))**.** *Let $|A| = m$ and $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_m$ be Condorcet domains on disjoint sets of alternatives $A, C_1, \ldots, C_m$, respectively. Then $\mathcal{D} = \mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)$ is again a Condorcet domain with*

$$|\mathcal{E}(\mathcal{D}_1, \ldots, \mathcal{D}_m)| = |\mathcal{E}| \prod_{i=1}^{m} |\mathcal{D}_i|. \tag{5}$$

Moreover, $\mathcal{D}$ is a symmetric domain if and only if all domains $\mathcal{E}, \mathcal{D}_1, \ldots, \mathcal{D}_m$ are symmetric.

A partial case of the above construction has already appeared in the literature (Raynaud, 1981; Fishburn, 2002; Danilov and Koshevoy, 2013). For the case $|A| = 2$, $\mathcal{E} = \{a_1a_2, a_2a_1\}$ and any two Condorcet domains $\mathcal{D}_1, \mathcal{D}_2$, the operation

$$\mathcal{D}_1 \star \mathcal{D}_2 \ := \ \mathcal{E}(\mathcal{D}_1, \mathcal{D}_2) \ = \ \{u_1u_2 \mid u_1 \in \mathcal{D}_1, \ u_2 \in \mathcal{D}_2\} \cup \{u_2u_1 \mid u_1 \in \mathcal{D}_1, \ u_2 \in \mathcal{D}_2\}$$

was used in Danilov and Koshevoy (2013) in order to construct a series of never-middle maximal Condorcet domains, namely,

$$a_1 \star a_2 \star a_3 \star \ldots \star a_n \tag{6}$$

with some parenthesization,[15] where a_i is identified with the trivial domain on a single alternative a_i. Let us call a domain *completely decomposable* if it is of the form (6) for some parenthesization. Evidently, every maximal completely decomposable domain is symmetric. For instance, we have

$$\begin{aligned}(a \star b) \star (c \star d) &= \{abcd, abdc, bacd, badc, cdab, cdba, dcab, dcba\},\\ a \star (b \star (c \star d)) &= \{abcd, abdc, acdb, adcb, bcda, bdca, cdba, dcba\}.\end{aligned}$$

These two domains turn out to be the two group separable maximal Condorcet domains on four alternatives (see the domains $\mathcal{D}_8$ and $\mathcal{D}_9$ and their embeddings in the 4-permutohedron in the appendix). In general, we have:

Theorem 4.1. *A maximal Condorcet domain is group separable if and only if it is completely decomposable.*

4.2 Indecomposable Domains

Danilov and Koshevoy (2013) discovered a series of symmetric maximal Condorcet domains that for any number of alternatives m have cardinality of just 4. Karpov and Slinko (2022b) call them *Raynaud domains* as Raynaud (1981) was the first who discovered such a domain in case of four alternatives calling it 'configuration K.' The (unique) Raynaud domain on four alternatives is given by the symmetric domain $\{abcd, bdac, cadb, dcba\}$ (see the domain $\mathcal{D}_7$ in the appendix).

Let $A = \{1, 2, \dots, m\}$ and define a permutation dk_m by

$$dk_m \;:=\; 24\cdots(2k)1(2k\pm 1)\cdots 53,$$

where $2k \pm 1$ is equal to $2k+1 = m$, if m is odd, and $2k-1 = m-1$ if m is even. For example, $dk_6 = 246153$ and $dk_7 = 2461753$. Denoting $e = 12\dots m$, Danilov and Koshevoy (2013) showed that the domains

$$\{e, \overline{e}, dk_m, \overline{dk}_m\}$$

are symmetric maximal Condorcet domains for every $m \geq 4$; moreover, these domains are evidently indecomposable.

The permutation dk_m is a special case of what is known as *simple permutation* (Albert and Atkinson, 2005).

Definition 18. *Let $i_1 i_2 \dots i_m$ be a sequence of distinct elements of $\{1, 2, \dots, m\}$. We say that a subsequence $i_k i_{k+1} \dots i_\ell$ is an* interval *of length ℓ in the sequence $i_1 i_2 \dots i_m$ if the set $\{i_k, i_{k+1}, \dots, i_{k+\ell-1}\} = \{a, a+1, \dots, a+\ell-1\}$ for some $a \in \{1, 2, \dots, m\}$. This interval is* trivial *if this subsequence has length 1 or m. A sequence without non-trivial intervals is called a* simple permutation.

[15] Note that the operation $\star$ is commutative but not associative.

For example, 21 is the only non-trivial interval in 521463, and 52463 is the only interval in 152463; the permutations 2413, 41352, 24153, 2475316, 24683157, and all permutations of the form dk_m are simple.

Definition 19. *For any permutation $u \neq e$, denote the domain $\mathcal{K}_u := \{e, u, \overline{u}, \overline{e}\}$, and say that $\mathcal{K}_u$ is a* Raynaud domain *if it is a maximal Condorcet domain.*

The following result highlights the role of simple permutations in our context.

Theorem 4.2 (Karpov and Slinko (2022b))**.** *A domain of the form $\mathcal{K}_u$ is a Raynaud domain if and only if u is a simple permutation.*

All Raynaud domains are indecomposable, and the following conjecture states that these are in fact the only indecomposable symmetric maximal Condorcet domains.

Conjecture 2. *Every indecomposable symmetric maximal Condorcet domain on $m \geq 4$ alternatives is a Raynaud domain.*

5 Large Condorcet Domains

As noted above, the main motivation for the work reported in Fishburn (1997, 2002) was the quest for large Condorcet domains. The title of Raynaud's paper from (1982) describes this motivation in more detail: 'The individual freedom allowed by the value restriction condition.' Thus, Raynaud (1982) derives the interest in large Condorcet domains from the goal to maximize individual preference freedom under the collective rationality constraint of a transitive majority relation.

For $m \leq 6$, Fishburn himself proved that F_m is indeed the Condorcet domain with maximal cardinality. Galambos and Reiner (2008) report that the same is true for $m = 7$. But Fishburn also showed that for sufficiently large m, the alternating scheme does not deliver the largest Condorcet domain; more concretely, he showed that for all $m \geq 16$ there are Condorcet domains that have a larger cardinality than F_m (Monjardet, 2009). Denoting by $f(m)$ the largest cardinality of any maximal Condorcet domain on m alternatives, Fishburn's result can be written as $f(m) > |F_m|$ for all $m \geq 16$.

However, the example produced by Fishburn used an operation on Condorcet domains similar to the one in Definition 16, in particular it was not a peak-pit domain that had cardinality larger than that of F_n. He thus posed the following modified question:

> Is it true that the alternating scheme domain is the largest in the class of connected domains with maximal width?

Galambos and Reiner (2008) gave an exact formula for the cardinality of Fishburn's domains; the following table list the first 13 values of $|F_m|$:

m	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert F_m \rvert$	1	4	9	20	45	100	222	488	1069	2324	5034	10840

They also showed that $f(7) = |F_7| = 100$ and emphasized the importance of Fishburn's conjecture. Monjardet (2009) denoted by $g(m)$ the size of the largest peak-pit Condorcet domain of maximal width so that Fishburn's conjecture could be written as: Is it true that $g(m) = |F_m|$?

The conjecture was finally refuted by Danilov et al. (2012) who showed that $g(42) > |F_{42}|$. The tool for the construction of an appropriate example was an operation that given two peak-pit domains of maximal width produces a larger Condorcet domain of the same class. Karpov and Slinko (2022a) improve their analysis by introducing a new construction called 'concatenation + shuffle scheme.' The advantage of this composition operation is that given two maximal peak-pit Condorcet domains with maximal width it produces another peak-pit Condorcet domain with maximal width that is again a *maximal* Condorcet domain (in contrast to the construction used in Danilov et al. (2012)). With the help of this construction Karpov and Slinko (2022a) show that already $g(34) > |F_{34}|$.

To date, the best known lower bounds for f and g are (Karpov and Slinko, 2022a):

$$\begin{aligned} g(m) &\geq 2.0767^m, \\ f(m) &\geq 2.2031^m. \end{aligned}$$

6 Conclusion

Let us conclude with some notes on what we have *not* covered here. First, in our treatment of the peak-pit domains we have left out some (non-elementary) facts and results on inversion triples, reduced decompositions and, more generally, the study of the so-called Bruhat lattice on the symmetric group of permutations; for a mathematically rigorous treatment, we refer the reader to Galambos and Reiner (2008); Danilov et al. (2012).

Second, we have confined ourselves to *maximal* Condorcet domains. While this class is arguably the most relevant and interesting class of Condorcet domains, some results hold more generally for the class of closed Condorcet domains. Most importantly, as shown in Puppe and Slinko (2019) every closed Condorcet domain (whether or not it is connected) naturally induces a median graph, and conversely every median graph defines (non-uniquely) a closed Condorcet domain. If a Condorcet domain is connected, the corresponding median graph is a subgraph of the permutohedron. Remarkably, some graphs induced by some closed Condorcet domains can never occur in the class of maximal Condorcet domains (for instance, trees that are not chains can be the induced median graphs of closed but not of maximal Condorcet domains, see (Puppe and Slinko, 2019, Th. 7)).

Third, the notion of Condorcet domain can be generalized to the case of weak orders, and even to partial orders. Indeed, a result analogous to Theorem 3.1 holds for weak orders (Puppe, 2018), and first steps towards an analysis of Condorcet domains of partial orders have been undertaken in Dittrich (2018).

References

Abello, J. and C. Johnson (1984): "How large are transitive simple majority domains?" *SIAM Journal on Algebraic and Discrete Methods*, 5, 603–618.

Albert, M. H. and M. D. Atkinson (2005): "Simple permutations and pattern restricted permutations," *Discrete Mathematics*, 300, 1–15.

Arrow, K. J. (1951): *Social choice and individual values*, New York: Wiley.

——— (1963): *Social choice and individual values, 2nd edition*, New York: Wiley.

Atkinson, M. D. and T. Stitt (2002): "Restricted permutations and the wreath product," *Discrete Mathematics*, 259, 19–36.

Ballester, M. and G. Haeringer (2011): "A characterization of the single-peaked domain," *Social Choice and Welfare*, 36, 305–322.

Björner, A., M. L. Vergnas, B. Sturmfels, N. White, and G. Ziegler (1999): *Oriented matroids*, Cambridge University Press.

Black, D. (1948): "On the rationale of group decision-making," *Journal of Political Economy*, 56, 23–34.

——— (1958): *The theory of committees and elections*, Cambridge: Cambridge University Press.

Craven, J. (1996): "Majority-consistent preference orderings," *Social Choice and Welfare*, 13, 259–267.

Danilov, V., A. Karzanov, and G. Koshevoy (2012): "Condorcet domains of tiling type," *Discrete Applied Mathematics*, 160, 933–940.

Danilov, V. and G. Koshevoy (2013): "Maximal Condorcet Domains," *Order*, 30, 181–194.

Dittrich, T. (2018): "Eine vollständige Klassifikation von Condorcet Domains für kleine Alternativenmengen," Ph.D. thesis, Karlsruhe Institute of Technology.

Elkind, E., P. Faliszewski, and P. Skowron (2014): "Characterizing the single-peaked single-crossing domain," in *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, 654–660.

Elnitsky, S. (1997): "Rhombic tilings of polygons and classes of reduced words in Coxeter groups," *journal of combinatorial theory, Series A*, 77, 193–221.

Felsner, S. (2012): *Geometric graphs and arrangements: some chapters from combinatorial geometry*, Springer Science & Business Media.

Fishburn, P. (1997): "Acyclic sets of linear orders," *Social Choice and Welfare*, 14, 113–124.

——— (2002): "Acyclic sets of linear orders: A progress report," *Social Choice and Welfare*, 19, 431–447.

Galambos, A. and V. Reiner (2008): "Acyclic sets of linear orders via the Bruhat orders," *Social Choice and Welfare*, 30, 245–264.

Gans, J. and M. Smart (1996): "Majority voting with single-crossing preferences," *Journal of Public Economics*, 59, 219–237.

Grandmont, J.-M. (1978): "Intermediate Preferences and the Majority Rule," *Econometrica*, 46, 317–330.

Guilbaud, G. T. and P. Rosenstiehl (1963): "Analyse algébrique d'un scrutin," *Mathématiques et Sciences Humaines*, 4, 9–33.

Humphreys, J. E. (1994): *Reflection Groups and Coxeter Groups*, Cambridge: Cambridge Univ. Press.

Inada, K. (1964): "A note on the simple majority decision rule," *Econometrica*, 32, 525–531.

Karpov, A. and A. Slinko (2022a): "Constructing Large Peak-Pit Condorcet Domains," *Theory and Decision*, in print.

——— (2022b): "Maximal Symmetric Condorcet Domains," *Order*, in print.

Kemeny, J. (1959): "Mathematics without numbers," *Daedalus*, 88, 577–591.

Kemeny, J. and L. Snell (1960): *Mathematical Models in the Social Sciences*, Ginn.

Leclerc, B. and A. Zelevinsky (1998): "Quasicommuting families of quantum Plücker coordinates," in *Kirillov's seminar on representation theory*, vol. 35, 85.

Leversidge, G. (2019): "On the number of Arrow single-peaked maximal Condorcet domains," *unpublished*.

Li, G., C. Puppe, and A. Slinko (2021): "Towards a classification of maximal peak-pit Condorcet domains," *Mathematical Social Sciences*, 113, 191–202.

Monjardet, B. (2007): "Condorcet Domains and Distributive Lattices," Lecture Notes, Ghent (personal communication).

——— (2009): "Acyclic Domains of Linear Orders: A Survey," in *The Mathematics of Preference, Choice and Order*, ed. by S. Brams, W. Gehrlein, and F. Roberts, Springer Berlin Heidelberg, Studies in Choice and Welfare, 139–160.

Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness," *Public Choice*, 35, 437–455.

Puppe, C. (2016): "The Single-Peaked Domain Revisited: A Simple Global Characterization," *KIT Working Paper Series in Economics 97.*

——— (2018): "The single-peaked domain revisited: A simple global characterization," *Journal of Economic Theory*, 176, 55 – 80.

Puppe, C. and A. Slinko (2019): "Condorcet domains, median graphs and the single-crossing property," *Economic Theory*, 67, 285–318.

Raynaud, H. (1981): "Paradoxical Results from Inada's Conditions for Majority Rule." Tech. rep., Stanford University Institute for Mathematical Studies in the Social Sciences.

——— (1982): "The Individual Freedom Allowed by the Value Restriction Condition." Tech. rep., Stanford University Institute for Mathematical Studies in the Social Sciences.

Roberts, K. (1977): "Voting over income tax schedules," *J. Public Economics*, 8, 329–340.

Rothstein, P. (1991): "Representative voter theorems," *Public Choice*, 72, 193–212.

Saporiti, A. (2009): "Strategy-proofness and single-crossing," *Theoretical Economics*, 4, 127–163.

Sato, S. (2013): "A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one," *Journal of Economic Theory*, 148, 259–278.

Sen, A. K. (1970): *Collective choice and social welfare*, San Francisco: Holden-Day.

Slinko, A. (2019): "Condorcet Domains Satisfying Arrow's Single-Peakedness," *Journal of Mathematical Economics*, 84, 166–175.

Slinko, A., Q. Wu, and X. Wu (2021): "A Characterization of Preference Domains that are Single-Crossing and Maximal Condorcet," *Economics Letters*, 204, in print.

Tohmé, F. and A. Saporiti (2006): "Single-Crossing, Strategic Voting and the Median Choice Rule," *Social Choice and Welfare*, 26, 363–383.

Zhan, P. (2022): "Sign representation of single-peaked preferences and Bruhat orders," *arXiv:2202.11352v1*.

Appendix

A All Maximal Condorcet Domains on Four Alternatives

In this section, we present the complete classification of all maximal CDs on four alternatives obtained by Dittrich (2018). Up to isomorphism or flip isomorphism there are exactly 18 different maximal CDs on the set $\{a, b, c, d\}$.

A.1 Connected Domains

There are in total six connected maximal CDs on four alternatives, all of them are peak-pit domains.

Connected Domains with Maximal Width

There are exactly three connected maximal CDs with maximal width:

$$\begin{aligned}
\mathcal{D}_1 &= \{abcd, abdc, acbd, bacd, badc, bdac, cabd, dbac\}, \\
\mathcal{D}_2 &= \{abcd, abdc, acbd, adbc, dabc, dbac, dbca\}, \\
\mathcal{D}_3 &= \{abcd, abdc, bacd, badc, bdac, bdca, dbac, dbca, dcba\}.
\end{aligned}$$

The first of these ($\mathcal{D}_1$) is the single-peaked domain with respect to the spectrum $d > b > a > c$, see Fig. 13 left; the second ($\mathcal{D}_2$) is a single-crossing domain with the pair *acbd* and *dbca* of completely reversed orders, see Fig. 13 middle; the third ($\mathcal{D}_3$) corresponds to Fishburn's alternating scheme and is the (uniquely) largest maximal CD on four alternatives with 9 members, see Fig. 13 right.

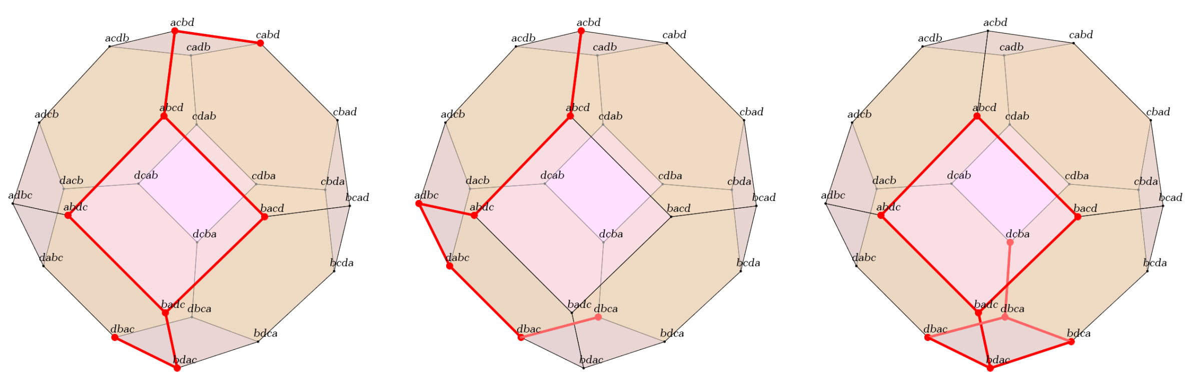


Figure 13: The three connected maximal CDs with maximal width $\mathcal{D}_1 - \mathcal{D}_3$

The characterizing never-conditions are,

$$\begin{aligned} \text{for } \mathcal{D}_1 &: \{aN_{abc}3, bN_{abd}3, aN_{acd}3, bN_{bcd}3\},\\ \text{for } \mathcal{D}_2 &: \{cN_{abc}1, bN_{abd}1, cN_{acd}1, bN_{bcd}3\},\\ \text{for } \mathcal{D}_3 &: \{bN_{abc}3, bN_{abd}3, cN_{acd}1, cN_{bcd}1\}. \end{aligned}$$

Connected domains without maximal width

There are exactly three connected maximal CDs without maximal width; they all have 8 members:

$$\begin{aligned} \mathcal{D}_4 &= \{abcd, abdc, acbd, acdb, bacd, badc, bcad, bcda\},\\ \mathcal{D}_5 &= \{abcd, abdc, acbd, adbc, bacd, badc, bcad, bdac\},\\ \mathcal{D}_6 &= \{abcd, abdc, acbd, adbc, bacd, badc, bcad, dabc\}. \end{aligned}$$

The first of these ($\mathcal{D}_4$) is flip isomorphic to an Arrow single-peaked domain, see Fig. 14 left; the second ($\mathcal{D}_5$) and third ($\mathcal{D}_6$) differ only by one order from each other, see Fig. 14 middle and right, respectively.

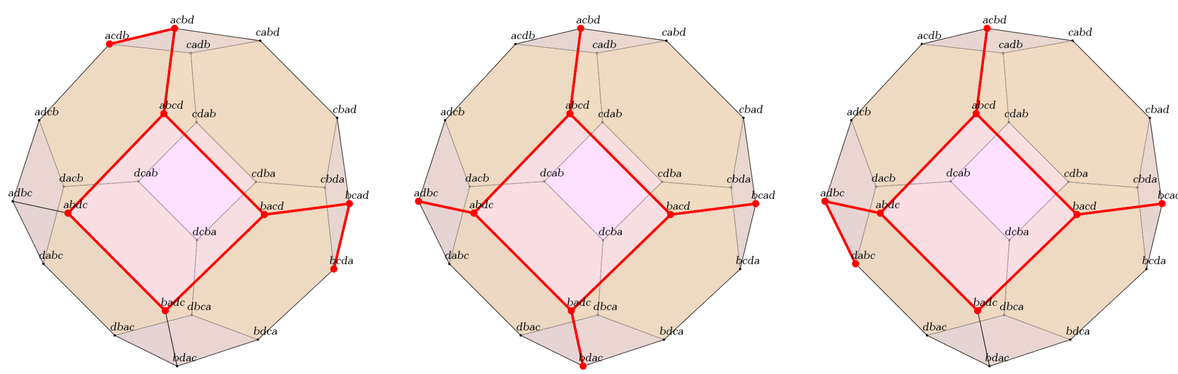


Figure 14: The three connected maximal CDs without maximal width $\mathcal{D}_4 - \mathcal{D}_6$

The characterizing never-conditions are,

$$\begin{aligned} \text{for } \mathcal{D}_4 &: \{cN_{abc}1, dN_{abd}1, dN_{acd}1, dN_{bcd}1\},\\ \text{for } \mathcal{D}_5 &: \{cN_{abc}1, dN_{abd}1, aN_{acd}3, bN_{bcd}3\},\\ \text{for } \mathcal{D}_6 &: \{cN_{abc}1, aN_{abd}3, aN_{acd}3, bN_{bcd}3\}. \end{aligned}$$

A.2 Symmetric Domains

The Indecomposable Raynaud Domain

The smallest maximal CD has four elements and is given by

$$\mathcal{D}_7 = \{abcd, bdac, cadb, dcba\}.$$

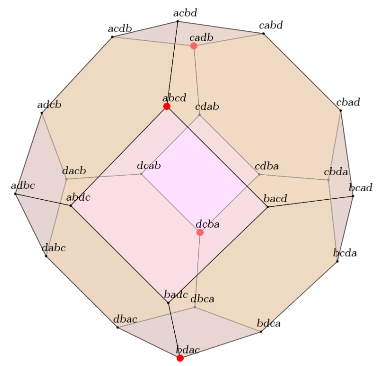


Figure 15: The smallest maximal CD with four elements

$\mathcal{D}_7$ is one of the type of maximal CDs with four members identified by Danilov and Koshevoy (2013) which exist for any number of alternatives; it is characterized by the following set of never-conditions:

$$\{cN_{abc}2,\ aN_{abd}2,\ dN_{acd}2,\ bN_{bcd}2\}. \tag{1}$$

Two Group Separable Domains

There are two group separable maximal CDs on a set of four alternatives, see Fig. 16:

$$\begin{aligned}
\mathcal{D}_8 &= \{abcd, abdc, bacd, badc, cdab, cdba, dcab, dcba\},\\
\mathcal{D}_9 &= \{abcd, abdc, acdb, adcb, bcda, bdca, cdba, dcba\}.
\end{aligned}$$

The characterizing never-conditions are,

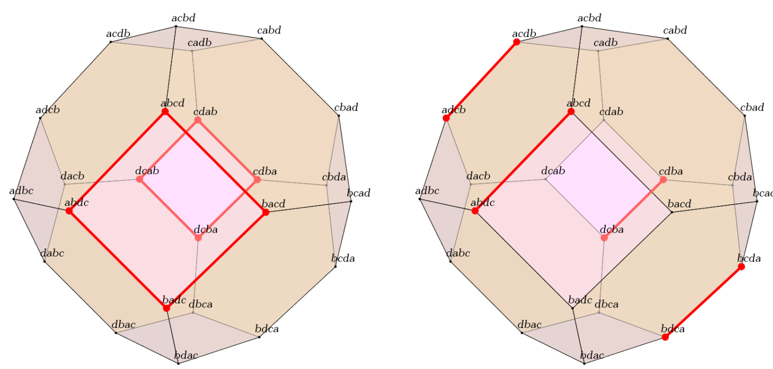


Figure 16: The two group separable maximal CDs $\mathcal{D}_8$ and $\mathcal{D}_9$

$$\begin{aligned}
\text{for } \mathcal{D}_8 &: \{cN_{abc}2, dN_{abd}2, aN_{acd}2, bN_{bcd}2\},\\
\text{for } \mathcal{D}_9 &: \{aN_{abc}2, aN_{abd}2, aN_{acd}2, bN_{bcd}2\}.
\end{aligned}$$

A.3 Mixed domains

Furthermore, there are nine 'mixed' domains, i.e., domains that are characterized by at least one never-middle condition *and* at least one never condition of another type. One of these domains ($\mathcal{D}_{10}$) has only seven members; all others ($\mathcal{D}_{11} - \mathcal{D}_{18}$ listed here in lexicographic order) have eight members, see Fig. 17.

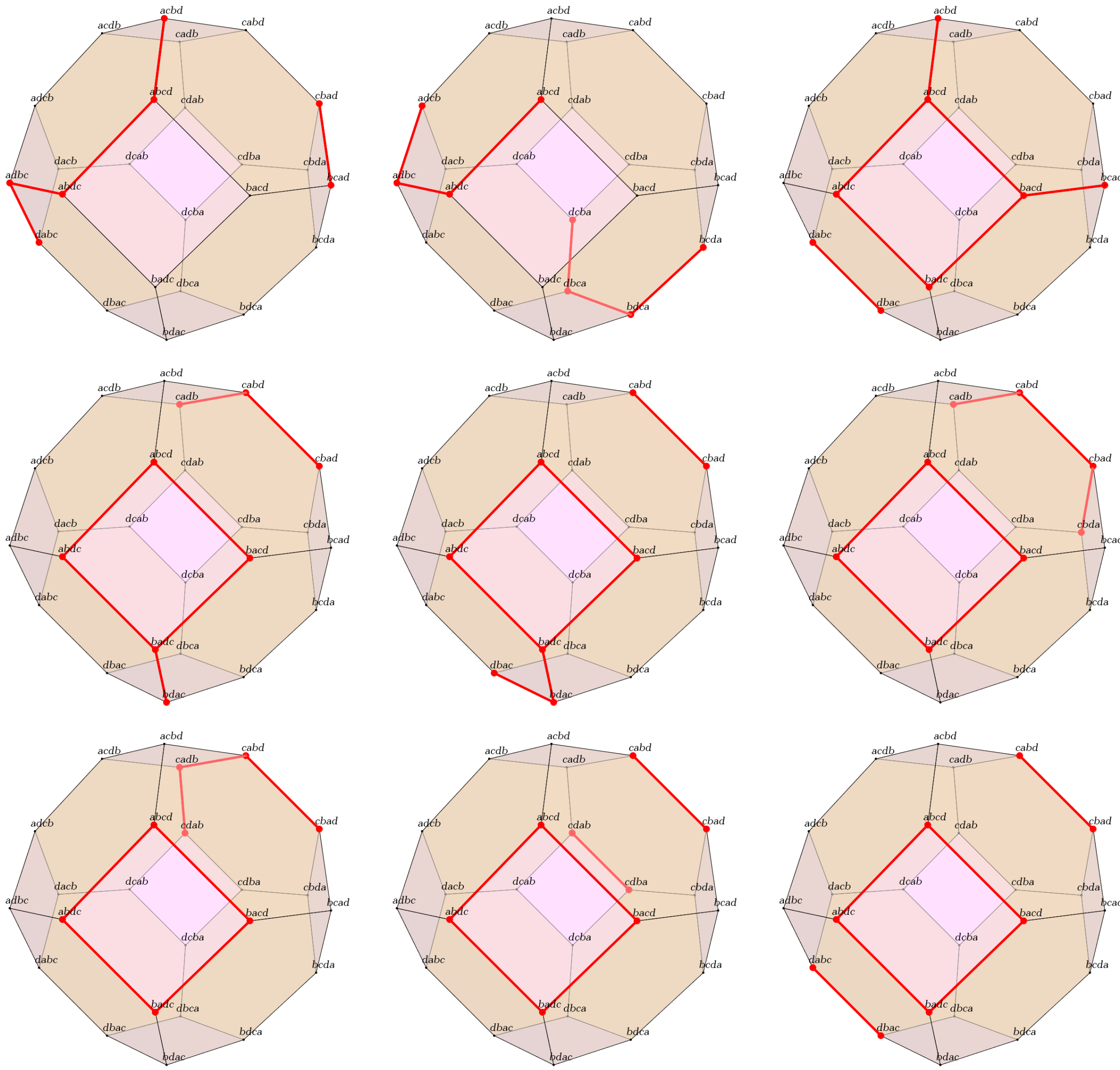


Figure 17: Mixed maximal CDs $\mathcal{D}_{10} - \mathcal{D}_{18}$ (top to bottom from left to right)

We have:

$$
\begin{aligned}
\mathcal{D}_{10} &= \{abcd, abdc, acbd, adbc, bcad, cbad, dabc\}, \\
\mathcal{D}_{11} &= \{abcd, abdc, adbc, adcb, bcda, bdca, dbca, dcba\}, \\
\mathcal{D}_{12} &= \{abcd, abdc, acbd, bacd, badc, bcad, dabc, dbac\}, \\
\mathcal{D}_{13} &= \{abcd, abdc, bacd, badc, bdac, cabd, cadb, cbad\}, \\
\mathcal{D}_{14} &= \{abcd, abdc, bacd, badc, bdac, cabd, cbad, dbac\}, \\
\mathcal{D}_{15} &= \{abcd, abdc, bacd, badc, cabd, cadb, cbad, cbda\}, \\
\mathcal{D}_{16} &= \{abcd, abdc, bacd, badc, cabd, cadb, cbad, cdab\}, \\
\mathcal{D}_{17} &= \{abcd, abdc, bacd, badc, cabd, cbad, cdab, cdba\}, \\
\mathcal{D}_{18} &= \{abcd, abdc, bacd, badc, cabd, cbad, dabc, dbac\}.
\end{aligned}
$$

The characterizing never-conditions are,

$$
\begin{aligned}
\text{for } \mathcal{D}_{10} &: \{aN_{abc}2, aN_{abd}3, aN_{acd}3, bN_{bcd}3\}, \\
\text{for } \mathcal{D}_{11} &: \{aN_{abc}2, aN_{abd}2, aN_{acd}2, cN_{bcd}1\}, \\
\text{for } \mathcal{D}_{12} &: \{cN_{abc}1, dN_{abd}2, aN_{acd}3, bN_{bcd}3\}, \\
\text{for } \mathcal{D}_{13} &: \{cN_{abc}2, dN_{abd}1, aN_{acd}3, dN_{bcd}1\}, \\
\text{for } \mathcal{D}_{14} &: \{cN_{abc}2, bN_{abd}3, aN_{acd}3, bN_{bcd}3\}, \\
\text{for } \mathcal{D}_{15} &: \{cN_{abc}2, dN_{abd}1, dN_{acd}1, dN_{bcd}1\}, \\
\text{for } \mathcal{D}_{16} &: \{cN_{abc}2, aN_{abd}3, dN_{acd}1, dN_{bcd}1\}, \\
\text{for } \mathcal{D}_{17} &: \{cN_{abc}2, dN_{abd}2, dN_{acd}1, dN_{bcd}1\}, \\
\text{for } \mathcal{D}_{18} &: \{cN_{abc}2, dN_{abd}2, aN_{acd}3, bN_{bcd}3\}.
\end{aligned}
$$

Working Paper Series in Economics

recent issues

No. 159 *Clemens Puppe, Arkadii Slinko:* Maximal Condorcet domains. A further progress report, December 2022

No. 158 *Daniel Hoang, Kevin Wiegratz:* Machine learning methods in finance: recent applications and prospects, December 2022

No. 157 *David Ehrlich, Nora Szech:* How to start a grassroots movement, September 2022

No. 156 *Klaus Nehring, Clemens Puppe:* Condorcet solutions in frugal models of budget allocation, March 2022

No. 155 *Clemens Puppe, Jana Rollmann:* Participation in voting over budget allocations. A field experiment, February 2022

No. 154 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 4/4. The third vote application, January 2022

No. 153 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 3/4. Tackling the Bundestag growth, January 2022

No. 152 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 2/4. Political spectrum, January 2022

No. 151 *Andranik S. Tangian:* Analysis of the 2021 Bundestag elections. 1/4. Representativeness of the parties and the Bundestag, January 2022

No. 150 *Marta Serra-Garcia and Nora Szech:* Choice architecture and incentives increase COVID-19 vaccine intentions and test demand, April 2021

No. 149 *Daniel Hoang, Fabian Silbereis and Raphael Stengel:* Do nonfinancial firms hold risky financial assets? Evidence from Germany, March 2021