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Structures of rational behavior in economics

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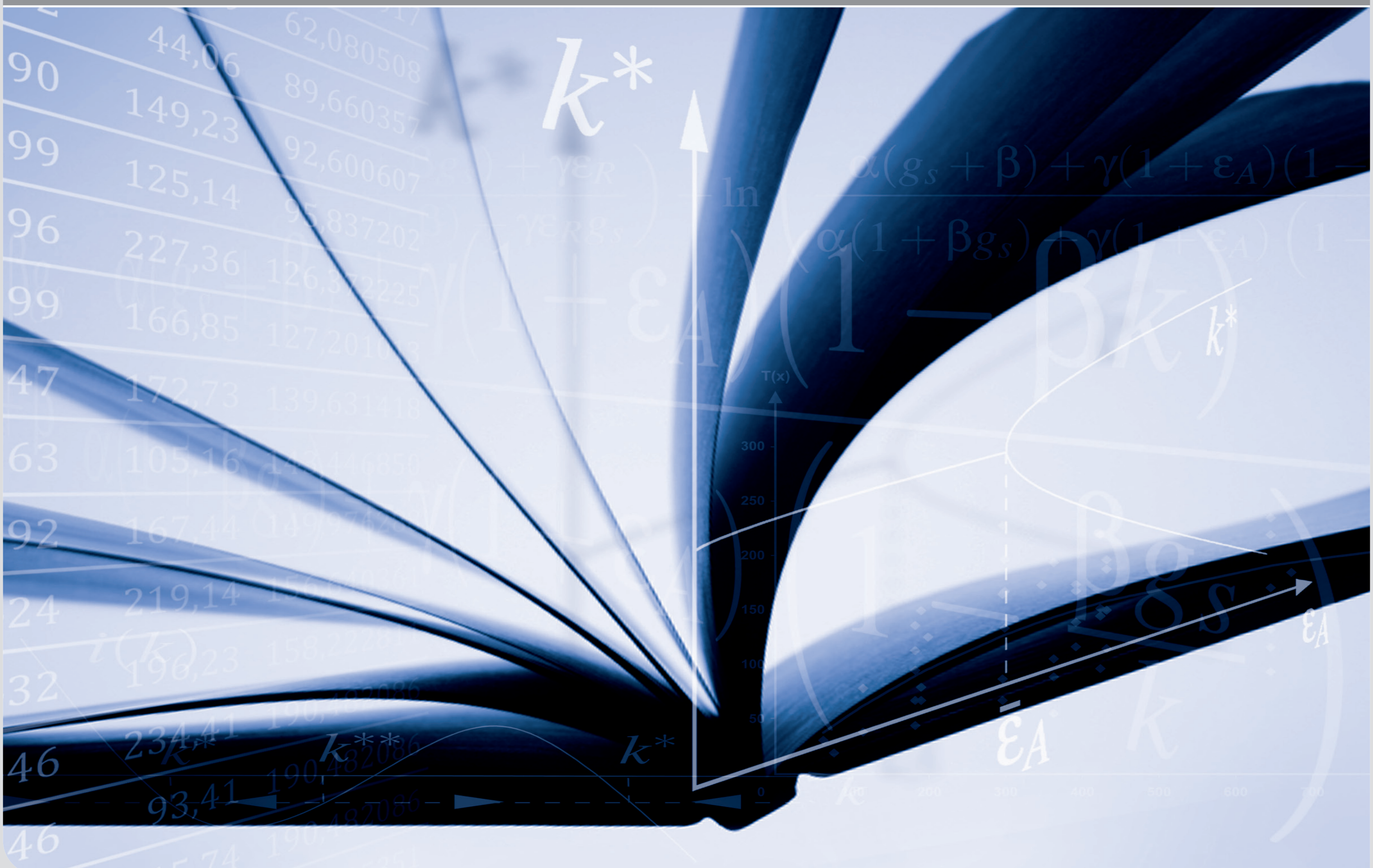

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Structures of Rational Behavior in Economics

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Abstract

Describing individual behavior, we will be concerned with an axiom system, which can be interpreted by Shephard's distance function. Based on this function, one can discover the individual's preference relation, from which the individual's demand function can be derived. We will realize that the axiom system describes rational behavior, satisfying the Strong Axiom of Revealed Preference. The axiom system presented in this article is closely related to a former one describing consumer behavior by income compensation functions. These different approaches will help to illuminate choice behavior from different points of view. We will also see that the axiom system presented in this article can be interpreted by the economic quantity index in welfare theory and by distance functions in producer theory.

JEL classification: D71, C78

Keywords: Economic Models, Demand Functions, Distance Functions, Rationality, Preferences, Producer Theory, Welfare Theory

1 Introduction

In this article, we will be concerned with an axiom system, which can be applied to various fields of economic activities.This axiom system will not only establish a convincing model of consumer behavior, but also an appropriate approach to quantity indices in welfare theory, and to the role of Shephard's distance function in producer theory. If the individual's behavior satisfies the axioms, then one can recover the individual's preference relation, which possesses all these properties usually assumed in economic theory. By means of this preference relation, the individual's demand function, which is rational with respect to the individual's preference relation and satisfies the Strong Axiom of Revealed Preference, can be derived.

We shall also add further results to the relationship of the above axiom system with another one based on income compensation functions, interpreting consumer behavior from another point of view [6].

Going further, it will be shown that the output distance function and the input distance function of production technologies also fulfill similar properties as required by the axioms. Therefore, similar structures in consumer and producer theory are generated.

The article proceeds in the following way. Firstly, the axiom system will be presented. Secondly, we will see that the economic quantity index fulfills the axioms. Thirdly, it will be shown that a reasonable theory of consumer behavior can be constructed as an interpretation of the axiom system. Then, we shall confine attention to a further axiom system concerned with the individual's compensated demand [6], showing that we can establish a close relationship between the two axiom systems.

Finally, we will consider the structure of input and output distance functions in producer theory which is also similar to the structure of the axiom system initially established.

2 A Model of Rational Behavior

In a previous article, a formal axiom system has been presented, which can be interpreted as a model of rational individual behavior in economics [6]. It does not assume that a utility function is given, but a preference relation is deduced from the individual's observable behavior.

This model is closely related to theories of demand such as the theory of revealed preference [14] and integrability theory [15], which start with demand functions describing consumer's observable behavior.

For modeling rational behavior, we will consider a set $X \subseteq \mathbb{R}^n_+$ of commodity bundles, and a function $D : X \times X \to \mathbb{R}_+$ describing behavior of an individual who reveals his or her preferences for commodity bundles compared to others.

By $x \in \mathbb{R}^n_+$, we mean $x_i \geq 0$ for all $i \leq n$, and by $x \in \mathbb{R}^n_{++}$, we mean $x_i > 0$ for all $i \leq n$.

The individual's behavior is supposed to fulfill the following axioms [8].

(E I) $D(x,x) = 1, \forall x \in X.$

(E II) For any $x', x'' \in X,$ and any fixed $x^0 \in X,$

(i) $D(x', x^0) = D(x'', x^0) \Rightarrow \forall x \in X : D(x', x) = D(x'', x),$

(ii) $D(x', x^0) > D(x'', x^0) \Rightarrow \forall x \in X : D(x', x) > D(x'', x).$

For any fixed $x^0 \in X :$

(E III) $D(x, x^0)$ is increasing in x,[1)]

(E IV) $D(x, x^0)$ is continuous in x,

(E V) $D(x, x^0)$ is concave in x, if X is convex.

$D(x, x^0)$ may be interpreted as the desirability of commodity bundle x relative to x^0 which is known through the agent's observable behavior. (E II) requires that the individual's evaluation is independent of the reference commodity bundle x^0. Therefore,

1) $D(x, x^0)$ is increasing, if for all $x^1, x^2 \in X : x^1 > x^2 \Rightarrow D(x^1, x^0) > D(x^2, x^0)$, where $x^1 > x^2$ means, $x^1_i > x^2_i, \forall i \leq n$.

if the individual appreciates x more than x' compared to the reference point x^0, then this appriciation must hold compared to any other reference commodity bundle. This is like measuring temperature by Celsius or Fahrenheit. (E III) to (E V) are regularity conditions.

In a previous article, it has been shown that Shephard's distance function $d(x,x') = \max\{\lambda \in \mathbb{R}_{++} | \frac{x}{\lambda} \succeq x'\}$ [16], where $\succeq$ is the agent's given relation, satisfies the conditions (E I) to (E V) under appropriate conditions [8, 7]. Therefore, $d(x,x')$ verifies the axiom system (E I) to (E V). The distance function can be applied in many ways in economics, for example, in production theory [1, 16], utility theory [3, 4], or welfare theory [2, 11].

In the next section, we will recognize that in welfare theory we can also observe a structure as described by (E I) to (E V).

3 Interpretations of (E I) to (E V) in Welfare Economics

Consider now a household that initially is observed to choose commodity bundle x^0 in price situation p. While the prices remain constant, the household is observed to choose another bundle x^1 at a later time. The quantities of goods will not have changed in the same way. The question arises whether or how much the household's welfare has changed. To answer this question, the quantity index

$$Q(p;x^1,x^0) = \frac{px^1}{px^0}, p \in \mathbb{R}^n_{++}, where\ px = \sum_{i=1}^{n} p_i x_i$$

can be used [3].

$Q(p;x^1,x^0)$ measures the increase or loss of welfare of a household when the household initially consumes commodity bundle x^0, and at a later time commodity bundle x^1. In most cases, the components of x^0 will not have risen or fallen by the same proportion, but generally, some components will have risen and others will have fallen. In order to find out the household's welfare change with the move from x^0 to x^1, one can multiply x^0 and x^1 with prices p^0 of the initial situation, where x^0 has been chosen, or with prices p^1 of the final situation, where x^1 has been chosen.

The corresponding quantity indices $Q(p^0; x^1, x^0) = \frac{p^0 x^1}{p^0 x^0}$ and $Q(p^1; x^1, x^0) = \frac{p^1 x^1}{p^1 x^0}$ are referred to as Laspeyres quantity index and Paasche quantity index, respectively. Both indices indicate whether the welfare of the household has risen or fallen. $Q(p; x, y)$ for any price level p are to be used to compare a large number of different "shopping baskets" on a consistent basis of prices $p \in \mathbb{R}^n_{++}$. It can be easily shown that $Q(p; x, y)$ satisfies (E I) to (E V), and is therefore a meaningful interpretation of the axiom system (E I) to (E V) in economics.

Theorem 1 *Let $X = \mathbb{R}^n_+$, and $p \in \mathbb{R}^n_{++}$, then the quantity index $Q(p; ., .); X \times X \Rightarrow \mathbb{R}$, $Q(p; x^1, x^0) = \frac{px^1}{px^0}$ satisfies (E I) to (E V) for $x^0 \in X \setminus \{0\}$.*

The proof follows immediately.

The preceding result also demonstrates that the axioms (E I) to (E V) are consistent.
In the next section, we will consider a consumer who behaves at the market in accordance with the hypotheses (E I) to (E V). It will be seen that we can develop a reasonable model of consumer behavior.

4 A Model of Consumer Behavior

As shown in [8], the axioms (E I) to (E V) can also be used to develop a model of consumer behavior. If the consumer's observable behavior adheres to the conditions (E I) to (E V), then his or her preference relation R_d can be deduced and possesses important properties. Based on R_d, one can construct a model of consumer behavior according to traditional consumer theory.
It is appropriate to define the preference relation R_d deduced from the consumer's behavior, as

Definition 4.1: For any reference commodity bundle x^0,
$xR_dx' :\Longleftrightarrow D(x,x^0) \geq D(x',x^0), \forall x,x' \in X.$

One can easily see that relation R_d possesses all qualities usually assumed in traditional consumer theory. This result has been shown in [8] and will be recalled in this article.

Theorem 2 *Let $X \subseteq \mathbb{R}^n_+$ be a closed cone [2] and assume (E I) to (E IV), then R_d is a complete [3], transitive, monotonic [4] and continuous [5] relation on X. If X is convex, then additionally assuming (E V), R_d is convex. Moreover, if more strictly*

(E V'): $D(x,x^0)$ is strictly concave,

holds, then R_d is strictly convex [6].

Based on Theorem 2, one can deduce a demand function describing consumer's behavior.

The individual's demand correspondence $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \longrightarrow 2^X$ usually is defined as a mapping assigning to every price-income situation (p,M) and budget set $B(p,M) = \{y \in X | py \leq M\}$ that set of alternatives $h(p,M)$ in $B(p,M)$, $h(p,M) \neq \emptyset$, which the individual chooses from $B(p,M)$. By definition, "h is rational" with respect to R_d, if for all $(p,M) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, $h(p,M) = \{x \in B(p,M) | \forall y \in B(p,M) : xR_dy\}$. This means that the individual's behavior corresponds to his or her preferences in every price-income situation. Note that $h(p,M)$ is single-valued, if (E V') instead of (E V) is assumed, and then h is called a demand function.

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[2] X is a cone in $\mathbb{R}^n_+$, if $x \in X \Rightarrow \lambda x \in X, \forall \lambda \geq 0$.

[3] A relation $\succeq$ is complete, if for all $x,y \in X, x \succeq y \vee y \succeq x$ holds.

[4] A relation $\succeq$ is monotonic, if for all $x^1,x^2 \in X : x^1 > x^2 \Rightarrow x^1 \succ x^2$, where $\succ$ denotes the asymmetric part of $\succeq$.

[5] A relation $\succeq$ on X is upper (lower) semicontinuous, if for all $x \in X$, the set $R(x) = \{y \in X | y \succeq x\}$ $(R^{-1}(x) = \{y \in X | x \succeq y\})$ is closed in X. If $\succeq$ is upper and lower semicontinuous on X, then $\succeq$ is continuous on X.

[6] $\succeq$ on the convex set $X \subseteq \mathbb{R}^n_+$ is strictly convex, if for all $x,y \in X$, and $(x \neq y)$, $x \succeq y$ implies $\lambda x + (1-\lambda)y \succ y, \forall \lambda \in (0,1)$.

In view of Theorem 2, the following Theorem 3 can be proven according to Proposition 2.1 in [9].

Theorem 3 *Let $X = \mathbb{R}^n_+$, and assume (E I), (E II), (E III), (E IV) and (E V′), then there exists a demand function $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ such that*

(a) h is rational with respect to R_d,

(b) h is homogeneous of degree 0 in (p, M), i.e. $h(\lambda p, \lambda M) = h(p, M), \forall \lambda > 0$,

(c) h is continuous in p,

(d) $\forall (p, M) \in \mathbb{R}^n_{++} \times \mathbb{R}_+ : ph(p, M) = M$.

Remark: If we, more weakly, assume (E V) instead of (E V′), then according to Theorem 1, R_d is convex and, therefore, $h(p, M)$ can be many-valued. It can then be demonstrated that $h(p, M)$ is upper hemicontinuous at every $p \in \mathbb{R}^n_{++}$. (For a proof, see [9, Proposition 3.2]).

The previous analysis has shown the surprising result that from the axioms (E I) to (E V′) a demand function can be derived describing consumer's behavior in accordance with traditional classical demand theory, where the agent's preference relation is given. We will also realize that the demand function derived in our model satisfies the Strong Axiom of Revealed Preference [10] being an important law in the theory of demand.

5 A Relationship to the Theory of Revealed Preference

The Theory of Revealed Preference has been pioneered by P. Samuelson [14]. It is a model of consumer behavior constructed by assuming the existence of a demand function describing individual's behavior which is in principle observable. An important hypothesis of this theory is the Weak Axiom of Revealed Preference [14] which we recall now.

Based on the given demand function $h : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$, the relation "revealed preferred", denoted by R, will be formulated in the following way [13, 14]: For $x \neq y$, define R as

$$xRy :\Leftrightarrow \exists (p, M) \in \mathbb{R}^n_{++} \times \mathbb{R}_+, x = h(p, M) \wedge y \in B(p, M).$$

By interpretation, xRy means that x is chosen in the price-income situation (p, M) where also y is available. The individual has shown by his or her behavior that he or she likes x more than y.

The Weak Axiom of Revealed Preference requires rational behavior of the agent. It will be stated here in the following version [13] :

For $x \neq y : xRy \Rightarrow \neg(yRx), \forall x, y$ in the range of h where $\neg$ means not.

Furthermore, we recall the relation "indirectly revealed preferred", R^*, which denotes the transitive hull of R. For $x \neq y$, define R^* as

$$xR^*y :\Leftrightarrow xRy \vee \exists x^1, ..., x^k : xRx^1 \wedge x^1Rx^2 \wedge ... \wedge x^kRy.$$

By interpretation, xR^*y means if x and y cannot be compared directly with each other in a price-income situation, then there exists a sequence of price-income situations and commodity bundles chosen in these situations which are revealed preferred to each other.

In order to accomplish Samuelson's revealed preference theory, Houthakker introduced the Strong Axiom of Revealed Preference [10].

The Strong Axiom of Revealed Preference can be formulated as [10, 13] :

For $x \neq y : xR^*y \Rightarrow \neg(yR^*x), \forall x, y$ in the range of h.

Theorem 4 *Let the hypotheses (E I) to (E IV) and (E V') hold. Then the demand function $h_d : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}^n_+$ being rational with respect to R_d satisfies the Strong Axiom of Revealed Preference.*

Proof. Consider $x, y \in \mathbb{R}^n_+$, $x \neq y$. In contrary, let us suppose that xR^*y and yR^*x hold. From xR^*y, we obtain $xRy \vee \exists x^1, ..., x^k : xRx^1 \wedge ... \wedge x^kRy$. If xRy, then by the definition of R, $\exists (p, M) : x \in h_d(p, M) \wedge y \in B(p, M)$. Since h_d is single-valued and rational with respect to R_d, we obtain xP_dy.
If xR^*y and $\neg xRy$, then by the same argumentation as previously, we obtain, $xP_dx^1 \wedge ... \wedge x^kP_dy$. In this case, transitivity and completeness of R_d imply xP_dy.
From yR^*x, we obtain yP_dx analogously. Therefore, we have $xP_dy \wedge yP_dx$. From this, xP_dx follows, a contradiction to the irreflexivity of P_d. ■

The preceding analysis has shown that the demand function h_d, deduced from the axioms (E I) to (E IV) and (E V'), which is rational with respect to R_d, is also rational in the sense of the Theory of Revealed Preference, because it satisfies the Strong Axiom of Revealed Preference.

6 A Relationship to Compensated Consumer Theory

In a former article [6], another model has been developed that is closely related to the axiom system (E I) to (E V). In the present article, we will point out further results.
Therefore, consider a closed set $X \subseteq \mathbb{R}^n_+$, $X \neq \emptyset$ of commodity bundles and a mapping $M(p, x)$ of $\mathbb{R}^n_{++} \times X$ into $\mathbb{R}_+$, where $p \in \mathbb{R}^n_{++}$ denotes price vectors and x commodity bundles. $M(p, x)$ is supposed to satisfy the following axioms [6]:

(C I) $\forall x \in X : [(\forall p \in \mathbb{R}^n_{++} : px \geq M(p,x)]$.

(C II) $\forall x,y \in X : [(x \neq y \wedge \forall p \in \mathbb{R}^n_{++} : px \geq M(p,y)) \Rightarrow \exists p' \in \mathbb{R}^n_{++} : p'y \leq M(p',x)]$.

(C III)(i) $\forall x,y \in X : [\exists p^0 \in \mathbb{R}^n_{++} : M(p^0,x) = M(p^0,y) \Rightarrow \forall p \in \mathbb{R}^n_{++} : M(p,x) = M(p,y)]$.

(C III)(ii) $\forall x,y \in X : [\exists p^0 \in \mathbb{R}^n_{++} : M(p^0,x) > M(p^0,y) \Rightarrow \forall p \in \mathbb{R}^n_{++} : M(p,x) > M(p,y)]$.

(C IV) $\forall p \in \mathbb{R}^n_{++}, \forall x \in X : [\exists z \in X : zRx$ and $pz = M(p,x)]$.

(C V) $M(p^0,x)$ is continuous in x.

(C VI) If X is convex, then $M(p^0,x)$ is strictly quasiconcave.

$M(p,x)$ has been interpreted as the income compensation function of an agent [6]. As verification, it has been shown that the well-known McKenzie income compensation function $m(p,x) = \min_{y \in X}\{py|y \succeq x\}$ [12], depending on the properties of a given preference relation $\succeq$, satisfies the axioms (C I) to (C VI) [8]. Furthermore, it has been shown that, based on the axioms (E I) to (E IV), the McKenzie income compensation function $m^0(p,x) = \min_{y \in X}\{py|yR_d x\}$ also satisfies the axioms (C I), (C III), (C IV), and (C V).

In order to demonstrate that (C II) also holds, we have to add a further condition which establishes a bridge from the axiom system (E I)-(E IV) to (C I)-(C V). This bridge is formed by assuming the following condition.

Property (α)

For all $x \in X$, there exists a $p \in \mathbb{R}^n_{++}$ such that $px = \min_{y \in X}\{py|D(y,x^0) \geq D(x,x^0)\}$ for any $x^0 \in X$.

Since $X \subseteq \mathbb{R}^n_+$ is assumed to be closed, and since by (E IV), $D(z,x^0)$ is continuous in z, $\min_{y \in X}\{py|D(y,x^0 \geq D(x,x^0)\}$ is well-defined. Therefore, in view of the definition of R_d, instead of (α) we can write

Property (α') For all $x \in X$, there exists a $p \in \mathbb{R}^n_{++}$ such that $px = \min_{y \in X}\{py|yR_d x\}$.

Property (α') means that for every x there exists a price situation p, where x is the cheapest commodity bundle which is at least as good as x itself. Applying Property (α'), we can prove the following theorem. Preliminarily, I will recall a result published in [7, Theorem 1, p.1228].

Lemma 1 *If $\succeq$ is a complete, transitive, continuous and locally nonsatiated [7] relation on a closed cone $X \subseteq \mathbb{R}^n_+$, then $m(p^0, x)$ is a continuous representation of $\succeq$ for any $p^0 \in \mathbb{R}^n_{++}$, and therefore, $m(p^0, x^1) \geq m(p^0, x^2) \Leftrightarrow x^1 \succeq x^2$.*

Theorem 5 *Let $X \subseteq \mathbb{R}^n_+$ be a closed cone and let (E I) to (E IV) hold, then*

(a) $m^0(p^0, x)$, for any $p^0 \in \mathbb{R}^n_{++}$, satisfies (C I), (C III), (C IV) and (C V).

(b) If, additionally, X is convex and (E V′) holds, then (C VI) is fulfilled.

(c) If, additionally, (α') is assumed, then (C II) is fulfilled.

Proof. According to Theorem 16 in [8], the validity of (C I), (C II), (C IV), and (C V) can be proven. Therefore, (a) is fulfilled. It remains to show (C II) and (C VI).
Proof for (b). According to Theorem 2, we already know that R_d is strictly convex if (E V′) instead of (E V) is assumed. Moreover, Theorem 2 and Lemma 2 yield that R_d is representable by $m^0(p^0, x)$. Therefore, it follows that $m^0(p^0, x)$ is strictly quasiconcave, and thus (C VI) is fulfilled.
Proof for c). Consider any $x, y \in X, x \neq y$, and assume
(1) $\forall p \in \mathbb{R}^n_{++} : px \geq m^0(p, y)$.
By contradiction, suppose, $\forall p' \in \mathbb{R}^n_{++} : p'y > m^0(p', x)$. In view of Property (α'), there exists $p^0 \in \mathbb{R}^n_{++}$ such that $p^0 y = m^0(p^0, y)$, and hence $m^0(p^0, y) > m^0(p^0, x)$. Since according to Theorem 2, R_d is a complete, transitive, continuous and monotonic relation on the closed cone X, it is also locally nonsatiated. Therefore, we can apply the above Lemma 1 and obtain $yP_d x$. From (1) and Property (α'), it also follows that there exists $\tilde{p} \in \mathbb{R}^n_{++}$, such that $\tilde{p}x = m^0(\tilde{p}, x)$ and $m^0(\tilde{p}, x) \geq m^0(\tilde{p}, y)$, and hence in view of the above lemma $xR_d y$. However, this is a contradiction to $yP_d x$, and therefore (C II)

[7] A relation $\succeq$ on a set X is locally nonsatiated, if for every $x \in X$ and every $\epsilon > 0$, there exists $y \in N_\epsilon(x) \cap X$ such that $y \succ x$, where $N_\epsilon(x)$ is the ϵ-neigborhood of x.

holds. ■

We thus have seen that if the income compensation function fulfills the axioms (E I) to (E V′) and (α'), it also fulfills (C I) to (C VI). It depends on the information which of these axiom systems should be applied to a real problem.

7 Structures in Producer Theory

Now, we will turn to producer theory, showing that we can discover similar mathematical structures underlying the economic model as in consumer theory. We will see that slight modifications of the axioms system (E II) to (E V) will be appropriate for modelling producer behavior. In producer theory, one studies the production plans of a firm. By P(x), we will denote the production possibility set of an input vector $x \in \mathbb{R}^n_+$ producing the output $y \in P(x) \subseteq \mathbb{R}^m_+$. P(x) is usually assumed to be a convex subset of $\mathbb{R}^m_+$. In producer theory, distance functions are also an important tool to describe the technology of a production process (see [1, 4], for instance). One example is the output distance function, defined as:

Definition 7.1: The function $D_0 : X \times Y \to \mathbb{R}_+$, where $X \subseteq \mathbb{R}^n_+$ and $Y \subseteq \mathbb{R}^m_+$,

$$D_0(x, y) = \inf\{\lambda > 0 | \tfrac{y}{\lambda} \in P(x)\},$$

is called the ″output distance function″, where λ is the smallest scalar by which we have to divide the output y and are still able to produce an output consistent with a given technology and a given input quantity described by the input vector x.

We will now characterize the output distance function of a technology by properties similar to the properties required by the axioms (E II) to (E V). Remember that x always denotes input and y denotes output.

$(P_o \text{ I})$ For any $x', x'' \in X$ and any $y^0 \in Y$,

$(i): \quad D_o(x', y^0) = D_o(x'', y^0) \Rightarrow \forall y \in Y : D_o(x', y) = D_o(x'', y),$

$(ii): \quad D_o(x', y^0) > D_o(x'', y^0) \Rightarrow \forall y \in Y : D_o(x', y) > D_o(x'', y).$

$(P_o \text{ II})$ $D_o(x, y^0)$ is decreasing on X for $y^0 \in Y$, *i.e.*

$x' > x'' \Rightarrow D_o(x', y^0) < D_o(x'', y^0), \forall x', x'' \in X.$

$(P_o \text{ III})$ $D_o(x^0, y)$ is increasing on Y for $x^0 \in X$, *i.e.*

$y' > y'' \Rightarrow D_o(x^0, y') > D_o(x^0, y''), \forall y', y \in Y.$

$(P_o \text{ IV})$ $D_o(x^0, y)$ is convex on Y for $x^0 \in X$, if Y is a convex set.

Moreover, it is also convenient sometimes to require that D_o is homogeneous of degree 1 in y, i.e.:
$(P_o$ V) For any $x^0 \in X : D_o(x^0, ty) = tD_o(x^0, y), \forall t > 0.$
Another characterization of output distance functions can be found in Färe [5, p. 30]. Note, since by $(P_o$ IV), $D_o(x^0, y)$ is convex on Y, then it is also continuous, if Y is convex and open.

We will now verify the axiom system $(P_o$ I) to $(P_o$ V) showing that the Cobb-Douglas technology with the production function $y = x_1^\alpha x_2^{1-\alpha}$ for $\alpha \in [0,1], y \in \mathbb{R}_{++}$, and $(x_1, x_2) \in \mathbb{R}^2_{++}$ satisfies these axioms.

Theorem 6 *The output distance function $D_o^C(x, y) = \inf\{\lambda > 0 | \frac{y}{\lambda} \leq x_1^\alpha x_2^{1-\alpha}, 0 \leq \alpha \leq 1\}$ of the Cobb-Douglas production technology satisfies (P_o I) to (P_o V).*

Proof. For $(P_o$ I) (i): Given $x', x'' \in \mathbb{R}^2_{++}$, and any $y^0 \in \mathbb{R}_{++}$. Assume $\lambda' = \inf\{\lambda > 0 | \frac{y_0}{\lambda} \leq (x_1')^\alpha (x_2')^{1-\alpha}\} = \lambda'' = D_o(x'', y^0)$.
Hence, $\lambda' = \frac{y^0}{(x_1')^\alpha (x_2')^{1-\alpha}} = \lambda'' = \frac{y^0}{(x_1'')^\alpha (x_2'')^{1-\alpha}}$, and thus, in view of $\lambda' = \lambda''$, for all $y \in \mathbb{R}_{++} : \frac{y}{(x_1')^\alpha (x_2')^{1-\alpha}} = \frac{y}{(x_1'')^\alpha (x_2'')^{1-\alpha}}$.
This immediately implies, $D_o^C(x', y) = D_o^C(x'', y)$ for $x', x'' > 0$.

For $(P_o$ I ii): the proof follows analogously.
Accordingly, the proofs for $(P_o$ III) to $(P_o$ V) also immediately follow. ■

By the above proof, we also have shown that the axiom system (P_o I) to (P_o V) is consistent.
Finally, we shall turn to the input distance function, having a similar structure as required by the axioms (E II) to (E V).

Definition 7.2: The function $D_i : Y \times X \to \mathbb{R}_+$, where $Y \subseteq \mathbb{R}^m_+$ and $X \subseteq \mathbb{R}^n_+$,
$D_i(y,x) = \sup\{\delta > 0 | \frac{x}{\delta} \in L(y)\}$,
where $x \in L(y) \iff y \in P(x)$, is called the "input distance" function.

For any given output y, $D_i(y,x)$ points out the fraction of input x which can still produce y. In analogy to the previous axiom system, we will now consider producer behavior based on the following postulates concerning the input distance function.

(P_i I) For any $y', y'' \in Y,$ and any $x^0 \in X,$
(i) $D_i(y',x^0) = D_i(y'',x^0) \Rightarrow \forall x \in X : D_i(y',x) = D_i(y'',x),$
(ii) $D_i(y',x^0) > D_i(y'',x^0) \Rightarrow \forall x \in X : D_i(y',x) > D_i(y'',x).$
(P_i II) $D_i(y^0,x)$ is increasing on X for $y^0 \in Y$.
(P_i III) $D_i(y,x^0)$ is decreasing on Y for $x^0 \in X$.
(P_i IV) $D_i(y^0,x)$ is concave on X for $y^0 \in Y,$ if X is a convex set.

Moreover, it is often assumed that D_i is homogeneous of degree 1 in x, i.e.,

(P_i V) $D_i(y^0, tx) = tD_i(y^0,x), \forall t > 0$ and $y^0 \in Y$.

We will now consider an example of an input distance function describing a technology of a plant producing two outputs with two inputs.
The input distance function is assumed to be
$D_i^p(y,x) = \frac{\sqrt{x_1 x_2}}{\sqrt{y_1^2+y_2^2}}$, for $x \in \mathbb{R}^2_{++}, y \in \mathbb{R}^2_{++}$.
Then we can see that the axioms (P_i I) to (P_i V) are satisfied.

8 Conclusion

In the preceding analysis, we have investigated the similarity of the underlying structures of different economic models. The axioms forming the frame of these models can meaningfully describe economic activities and techniques. We have studied axiom systems which can be interpreted reasonably by index numbers in welfare theory, and in consumer and producer theory. Pointing out the similarities facilitates the analysis of economic behavior.

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