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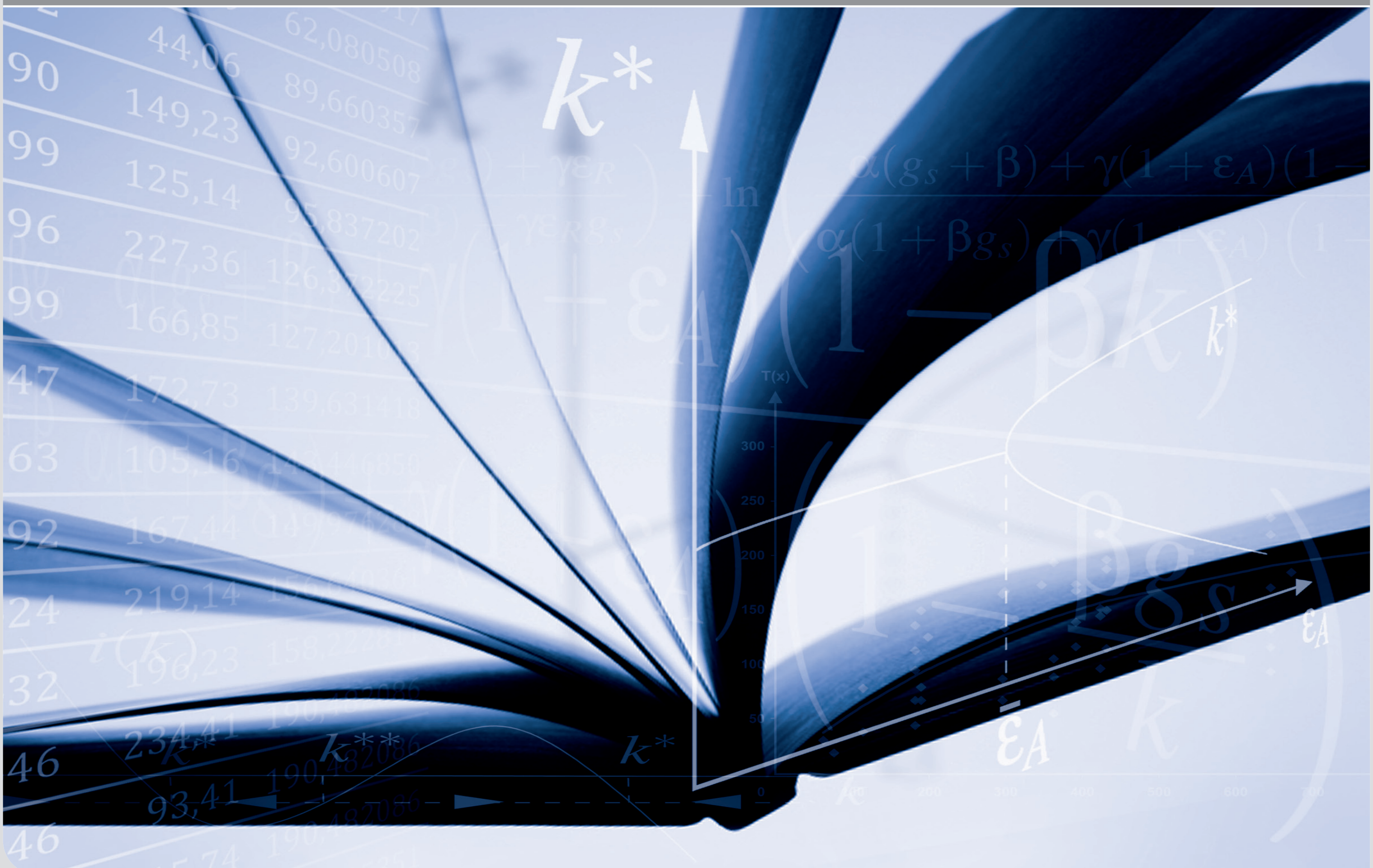

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Intermodal Competition between Intercity Buses and Trains – A Theoretical Model

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Abstract

The intercity bus market in Germany was deregulated in 2013. As a consequence, there is now a dense network of intercity bus lines. For the first time, the German state-owned railway company Deutsche Bahn AG faces intermodal competition in public intercity passenger land transport on a large number of lines. This paper examines market entry factors for intercity bus companies and price reactions of the incumbent railway company from a theoretical perspective. Our model builds on Salop's circular city model to describe the horizontal product differentiation among the bus companies. At the same time, the railway company occupies the center of the circle and offers a higher product quality than the buses. It dominates the market, while a number of bus companies constitute an oligopolistic competitive fringe. In the subsequent comparative statics analysis, it is shown that the quality differential between the train and bus services have a considerable effect on market entry decisions by buses as well as on price reactions by the incumbent railway company. In particular, on routes where the quality advantage of railway services is rather small, buses are more likely to enter and the railway company will respond with a stronger price reduction than on other routes.

Keywords: intermodal competition, intercity railway and bus services, Salop circle model with center, vertical and horizontal product differentiation, dominant firm with oligopolistic fringe

JEL codes: R40, L11, L13

1 Introduction

For decades, the market for public intercity passenger land transport in Germany was characterized by a monopolistic structure and dominated by the state-owned railway company Deutsche Bahn AG. The reason is that intercity bus transports were virtually prohibited. While there is still very limited intramodal competition in intercity railway transport[1], the market for intercity bus transport was successfully deregulated in 2013. As a result, numerous bus companies such as Flixbus, MeinFernbus, Postbus and Megabus have established their services in the subsequent years.[2]

By now, there is a nation-wide network of intercity bus lines in Germany. The number of intercity bus passengers has increased from 3 million passengers in 2012 up to 22.8 million passengers in 2017 (Statistisches Bundesamt (2018)). At the same time, the market share of intercity buses in public intercity passenger

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[1]The incumbent Deutsche Bahn AG has a market share of 99% in railway long-distance services (Bundesnetzagentur 2018).

[2]Beginning in 2015, the bus company Flixbus has acquired more and more of the other bus companies and now reached a marked share of more than 95%. Most of the remaining intercity bus transport is run by the railway company Deutsche Bahn AG, the French bus operator BlaBlaBus and by the Czech railway operator Regiojet.

transport (including air transport) has increased from 2% in 2012 up to 12% in 2017 in terms of passenger kilometers.[3] As a consequence, for the first time Deutsche Bahn AG faces intermodal competition in public intercity passenger land transport on a large number of lines.[4]

This paper provides a theoretical model of the particular competitive situation in the German long-distance land transport market during the years 2013 to 2015, when intercity bus transport was characterized by a considerable degree of intramodal competition, alongside with the intermodal competition between the railways and the buses. The model provides the background for empirical studies on the sector that have been published separately, see Gremm (2018) and Gremm (2019).

The German intercity bus market is characterized by low prices. While intercity bus tickets range between 5 and 7 euro cents per kilometer, even the special prices of trains[5] are about twice as high with 12 to 16 euro cents per kilometer.[6] Thus, price sensitive customers have partly moved from train to bus services. How many of the bus passengers actually came from the railways is a question of some political importance; therefore, depending on the respective survey, some estimates range between 10 and 15% (Bundesamt für Güterverkehr 2015) while others go up to 55%.[7] For these reasons, it is expected that the new market for intercity bus services is going to have some effect on the existing railway services. Some response to these new competitors by Deutsche Bahn AG could be observed in 2014 and 2015. It launched several additional special price campaigns, it abstained from increasing its normal ticket prices for the second class, and it announced a long-term customer campaign (Deutsche Bahn AG (2015)).

Customers evaluate railway transport generally as being of higher quality than bus transport. The most important reason is that, on average, a bus ride takes about 47% more time than the train (see Gremm (2019)). Moreover, trains are generally more comfortable than buses. However, when intercity buses offered internet access to their passengers (since they address students and other young people in particular) Deutsche Bahn hastened to improve its onboard internet access in trains as well after 2013.

Note further that the total capacity of train transport and the density of the long-distance train network exceed those of intercity bus transport by far. Consistently, the market shares in terms of passenger kilometers, given above, imply a market share relationship between trains and buses of 6 to 1 (Statistisches Bundesamt (2018)). Thus, it seems adequate to interpret the relationship of Deutsche Bahn AG to the intercity bus companies as one of an incumbent dominant firm (railway company) with an oligopolistic fringe of entrants (buses). This characterization holds for the market situation in Germany between 2013 and 2015 when there was still considerable intramodal competition in the intercity bus market.

Our model assumes that the competition between intercity buses and intercity trains is, on the one hand, affected by intermodal competition between high quality trains and lower quality buses and, on the other hand, by intramodal competition between the intercity bus companies that offer basically the same quality, but in different varieties. We describe this complex situation by a model of two-dimensional product differentiation. It is assumed that intercity trains in general offer a better service compared to intercity buses. This refers to a situation of vertical product differentiation, where consumers have the same preference order regarding a product's characteristic, which therefore can be called 'quality'. At the same time, the various intercity bus companies offer products of the same quality, but differ in the 'variety' of their respective products. In this horizontal product differentiation, consumers have different preference orderings over a characteristic, which consequently means that in the case of equal prices, consumers do not all buy the same variety of the product.

Our model builds on Salop's circular city model to describe the horizontal product differentiation among

[3] According to own calculations based on Statistisches Bundesamt (2018).

[4] For routes of more than 500 kilometers between certain large cities, there is also competition by air traffic, since the aviation market was already deregulated in 1993. With 13%, the market share of air transport is comparable to that of intercity buses (Statistisches Bundesamt (2018)).

[5] As part of a yield management system, Deutsche Bahn AG offers several tickets with especially low prices.

[6] Based on own calculations, data collected two weeks before departure in June 2015.

[7] According to a survey of Verkehrsclub Deutschland e.V. (Bundesamt für Güterverkehr (2016)).

the bus companies. These bus companies incur fixed cost if they enter the market, so that the number of firms on the oligopolistic fringe is determined by the zero profit condition. At the same time, the vertically differentiated railway company occupies the center of the circle. Consumers are therefore described by their willingness to pay for quality and their preference for the varieties offered by the bus companies, which leads us to Economides' two-dimensional cylinder consumer space model. We provide two versions of the model, a general one and a simplified one. In the general model, consumers' strength of preference for vertical quality is continuously distributed on the $[0, 1]$ interval. The model can be solved, but it leads to a complicated description of equilibrium market shares which is not well suitable for comparative statics analyses.

This paper aims to contribute to the understanding of intermodal competition between intercity buses and trains and to produce testable hypotheses as basis for empirical studies. In particular, we want to answer the question which factors influence intercity bus companies' decisions to enter the market and start their business, considering an existing railway system. Furthermore, we are interested in the price reactions of the incumbent railway company to the buses. In the simplified model, consumers' strength of preference for vertical quality assumes just three distinct values, and comparative statics of all kinds can be derived. For example, we show that changes in railway's service quality have a crucial effect on pricing behavior and on buses' decisions of market entry. We regard each transport relation between one city and another one as a separate market, so that each relation constitutes a separate case of application of our model. This cross-sectional view has also been applied in the empirical analyses by Gremm (2018) and Gremm (2019).

The next section describes the theoretical developments in the literature, on which we build our model or to which it is related. Section 3 presents the general model, followed by a simplified version of the model in Section 4. Building on the comparative static analyses of the simplified model (Section 4.5), Section 5 discusses the empirical hypotheses for the German land transport market that have been addressed in the studies mentioned above. Finally, in Section 6 we conclude the results of this paper. Longer proofs are set out in the Appendix.

2 Related work

In the literature, different approaches have been used to model intermodal competition. Frequently, discrete choice models are applied to describe the demand in intermodal as well as intramodal competition in transport. The discrete choice model by Ben-Akiva and Lerman (1985) laid the ground for subsequent research work by Ivaldi and Vibes (2008), Mancuso (2014), Niedhart (2009) or Preston, Whelan and Wardman (1999). For example Ivaldi and Vibes (2008) simulate intermodal and intramodal competition between aircraft, car and railways, whereby consumers can additionally choose between different airlines. For this purpose, they calibrate a nested logit demand model; on the supply side they assume Bertrand-Nash competition. Mancuso (2014) expands on the simulation model of Ivaldi and Vibes (2008) and applies it to the route between Milan and Rome to simulate the competition between trains and planes. A number of other papers use the discrete choice approach to simulate different market entry strategies in intermodal or intramodal competition and to analyse their effects on the incumbent, see for example Johnson and Nash (2012), Niedhart (2009), Preston, Whelan and Wardman (1999), Raturi et al. (2013), Wang and Yang (2005).

Another strand of literature on intermodal competition uses vertical product differentiation models, see Hsu, Lee and Liao (2010), Hsu and Lee (2014), Raturi et al. (2013), Wang and Yang (2005), Yang, Kong and Meng (2001), or Yang and Zhang (2012). Vertical product differentiation can also be coupled with the assumption of a dominant position of the supplier of the highest quality. In a non-transport context, such a market situation has been examined by Balan and Deltas (2013, 2014) as well as by Matsushima and Liu (2012). In these papers, a dominant firm offering a high quality is contested by a competitive fringe of several other companies producing a lower quality. Since the low quality products are assumed to be homogeneous in these papers, there is perfect competition within the competitive fringe.

In contrast, we present a model which includes vertical and horizontal product differentiation. Neven and Thisse (1990) extend Hotelling's linear market to a quadratic market to combine both types of product differentiation. In a similar way, Economides (1993) presents a model of horizontal and vertical product differentiation, in which the consumers and producers are situated on a cylinder. This model is built on Salop (1979), who was the first to modify Hotelling's linear market to a circular market. As in Economides (1993), the model presented in this paper adds a vertical component to the Salop circle, so that we arrive at a cylinder model. However, in contrast to Economides (1993), we also consider a company in the center of the cylinder, at the top. This is the railway incumbent, whereas the competing bus companies are situated on the perimeter, at the bottom of the cylinder. They constitute an oligopolistic fringe.

There are some research papers extending the Salop circle model by introducing a center. Heal (1980) describes a market in which the producer of a good is situated at the center of the circle while there are several retailers on its perimeter. Although the context is different, Heal (1980) already mentions the possibility to introduce different kinds of product differentiation to this setting. In a more related context, Balasubramanian (1998), Bouckaert (2000), Cheng and Nault (2007), and Madden and Pezzino (2011) use the circular market with center. In these models, the center is occupied by an internet or mail order business, whereas the physically reachable purchasing opportunities are spread on the circle perimeter. If customers buy from one of the retailers, they have to incur transport costs that depend on their individual positions on the circumference whereas, if they buy from the center, there are fixed shipping costs.

To our knowledge, the Economides (1993) cylinder model has not been extended by a company in the center so far. Moreover, the market situation of a dominant firm with a competitive fringe has so far always been modeled with the assumption that the fringe companies are in perfect competition. Our model, therefore, incorporates two novel aspects that allow us to model a dominant firm offering the high quality of the good and competing with an oligopolistic fringe of firms offering different varieties of a lower quality of the good.

3 The general model

We consider a circular market with center. The length of the circumference of the circle is equal to one. The bus operators are situated on its perimeter, while the railway company occupies the center. The product sold in the market can differ both in its (vertical) quality and its (horizontal) variety. The railway company offers the high quality product q_h, the bus operators each sell one different variety of the low quality version q_l, in which $0 \leq q_l < q_h$. There are $n \geq 2$ bus operators in the market. The position k_j of bus operator $j \in [1, n]$ on the circle indicates its respective variety. The bus operators are spread equidistantly on the perimeter, therefore is $k_j = \frac{j-1}{n}$. This means, there is a maximal degree of horizontal product differentiation amongst the provided varieties.

On the demand side, each consumer i is characterized by two properties: Firstly, his or her position on the circle $k_i \in [0, 1)$; secondly, his or her preference for quality $\theta_i \in [0, 1]$. Therefore, we can describe the consumer space as a cylinder sitting on top of the circle (which is the product space) with θ displayed on the vertical axis. It then can be cut open and rolled out to get a two-dimensional consumer space with length and height both equal to one. For simplification, the mass of consumers is normalized to one and the consumers are uniformly and independently distributed in the two dimensions.

Consumers purchase exactly one unit of the product. Their willingness to pay depend on their preferences for quality and variety and the respective qualities and varieties of the products. In addition, there is a basic utility V_0 which is assumed to be large enough so that every consumer buys. Willingness to pay for a bus decreases with the distance on the circle perimeter between the consumer and the bus company where he or she buys. These horizontal distances are weighted with a factor $t > 0$ which can be interpreted as the intensity of the preference for a certain variety of the product. Willingness to pay for the railway trip is reduced by tR, where R is a fixed parameter identical for all consumers. Note that $R \geq 0$ is not in a geometrical sense the distance between the perimeter and the center of the circle (the radius). It is simply the distance to the railway company that is equal for all consumers and hence independent of the circle's circumference. The utility of consumer i, described by (k_i,θ_i), when choosing either the railway company h or one of the bus operators j, is shown below.

$$U_i = \begin{cases} V_0 + \theta_i \cdot q_l - p_j - t \cdot |k_i - k_j| & \text{when choosing bus operator } j \text{ with price } p_j \\ V_0 + \theta_i \cdot q_h - p_h - t \cdot R & \text{when choosing railway company } h \text{ with price } p_h \end{cases}$$

Thus, while the distance cost tR to the railway company is identical for all consumers, the impact of railway quality on willingness to pay depends on consumer i's individual preference for quality, $\theta_i \cdot q_h$. In contrast, buses are not differentiated with respect to their quality (q_l identical for all j) but by their locations relative to the consumers' locations.

Note that the term tR counteracts railway's general advantage in the vertical product differentiation. In the spirit of this model, vertical product differentiation should be the dominating effect in the large. At least, it is assumed throughout that $tR < q_h - q_l$, so that consumers with $\theta_i = 1$ will prefer the railways if prices are the same. Stricter assumptions could also be made, e.g. $2tR < q_h - q_l$, so that this property holds also for the average θ-type. However, unless $R = 0$, there are always some consumers who do not prefer the railways if prices are the same, particularly those with $\theta_i = 0$ and $k_i = k_j$ for some j.

We consider a game in two stages. In the first stage, bus operators can choose whether to enter the market. They enter as long as their expected profits exceed the fixed entry costs F. The center of the circle is always taken by the railway company. In the second stage, the market participants compete in their prices. We seek a subgame perfect Nash-equilibrium. As per backward induction, we start with the price competition. At first, however, we need to take a closer look at consumers' demand behavior in order to determine price decisions

3.1 Demand in the general model

In this model, consumers have to choose between three possibilities. They can buy the product from the railway company in the center or from the bus operator situated in front or behind them on the circle perimeter.[8] Consider consumers who are indifferent between two adjacent bus operators j and $j+1$. Their horizontal position, i.e. their value of k, is independent of their preference for (vertical) quality since all bus operators offer the same quality q_l. So, the position $k_{j,j+1}$, which is the position between bus j and $j+1$ on the perimeter, in which all consumers $(k_{j,j+1},\theta)$ are indifferent between these two alternatives, can be calculated from the above utility function as follows.

$$\begin{aligned} & V_0 + \theta q_l - p_j - t(k_{j,j+1} - k_j) = V_0 + \theta q_l - p_{j+1} - t(k_{j+1} - k_{j,j+1}) \\ \Leftrightarrow \quad & k_{j,j+1} = \frac{1}{2}\left(\frac{p_{j+1} - p_j}{t} + k_j + k_{j+1}\right) \end{aligned} \tag{1}$$

Let us furthermore assume that, for each j, the consumer $(k_{j,j+1},0)$, who is indifferent between the two buses and who has the least possible preference for quality, strictly prefers bus j over the railway company h. In this case there is a value $\theta_1^{j+} > 0$ such that all consumers with $\theta \leq \theta_1^{j+}$ who are situated between k_j and $k_{j,j+1}$ prefer j over h (and let θ_1^{j+} be the greatest value for which this holds). This means, consumer $(k_{j,j+1},\theta_1^{j+})$ is indifferent between j and h:

$$\begin{aligned} & V_0 + \theta_1^{j+} q_l - p_j - t(k_{j,j+1} - k_j) = V_0 + \theta_1^{j+} q_h - p_h - tR \\ \Leftrightarrow \quad & \theta_1^{j+} = \frac{p_h - p_j + tR - t(k_{j,j+1} - k_j)}{q_h - q_l} \end{aligned}$$

[8] In the proof set out in Appendix A.3 we will also have a glance at the possibility of buying at bus operator that is further away, since we can show that this will not happen in equilibrium.

Similarly, we also assume that, for each j, the consumer $(k_j, 1)$, who has the highest possible preference for quality and the same horizontal position as bus j, strictly prefers the railway company h over bus j. Then there is a value $\theta_2^j < 1$ such that all consumers with $\theta > \theta_2^j$ prefer h over j (and let θ_2^j be the lowest value for which this holds). This means that consumer (k_j, θ_2^j) is indifferent between h and j:

$$\begin{aligned} & V_0 + \theta_2^j q_l - p_j = V_0 + \theta_2^j q_h - p_h - tR \\ \Leftrightarrow \quad & \theta_2^j = \frac{p_h - p_j + tR}{q_h - q_l} \end{aligned}$$

Currently, we rely on the preliminary assumptions that $\theta_1^{j+} > 0$ and $\theta_2^j < 1$ hold for all j. One can easily see that $\theta_2^j \geq \theta_1^{j+}$ holds. Using these critical values, for any given $k_i \in [k_j, k_{j,j+1}]$, the vertical part of the consumer space can be divided into three regions, as illustrated in Figure 1. Consumers with $\theta > \theta_2^j$ will choose the railways, irrespective of their particular θ–level. Consumers with $\theta < \theta_1^{j+}$ will choose bus operator j, irrespective of their particular θ–level. Particularly interesting is therefore the choice of consumers with preference for quality between θ_1^{j+} and θ_2^j. They choose the railway company h or bus j depending on, both, their θ_i and their horizontal position k_i on the circle perimeter.

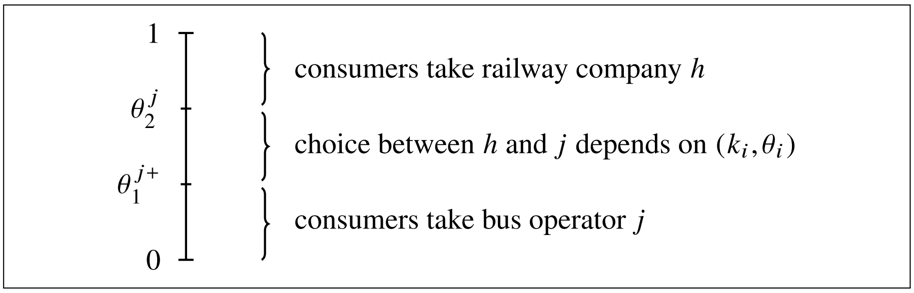

Figure 1: Vertical separation of the consumer space for any $k_i \in (k_j, k_{j,j+1})$

A consumer (k_i, θ_i) with $\theta_1^{j+} \leq \theta_i \leq \theta_2^j$ is indifferent between the railway and bus j when the following equation holds.

$$V_0 + \theta_i q_h - p_h - tR = V_0 + \theta_i q_l - p_j - t(k_i - k_j)$$

By rearranging terms, we can determine a function θ dependent on k, denoted by $\theta_{j+}(k)$. Every consumer on this function is indifferent between h and j:

$$\theta_{j+}(k) = \frac{p_h - p_j + tR - t(k - k_j)}{q_h - q_l} \qquad \text{for} \quad k \in [k_j, k_{j,j+1}]$$

Up to this point, we have only examined the section of the consumer space between k_j and $k_{j,j+1}$. Turning to the section from $k_{j,j-1}$ to k_j, a critical θ_1^{j-} can be determined for these consumers analogously to the above: $\theta_1^{j-} = \frac{p_h - p_j + tR - t(k_j - k_{j-1,j})}{q_h - q_l}$. Based on this, a similar picture as in Figure 1 could be drawn for the subspace $[k_{j-1,j}, k_j]$. For the consumers with $\theta_1^{j-} \leq \theta_i \leq \theta_2^j$, we can analogously derive the function

$$\theta_{j-}(k) = \frac{p_h - p_j + tR - t(k_j - k)}{q_h - q_l} \qquad \text{for} \quad k \in [k_{j-1,j}, k_j]$$

The functions $\theta_{j+}(k)$ and $\theta_{j-}(k)$ can be combined to the function $\theta_j(k)$ defined for $k \in [k_{j-1,j}, k_{j,j+1}]$. In the shared boundary point (k_j), both parts have the same value, namely θ_2^j. In the distant boundary points, the functions attain their respective value of θ_1 (θ_1^{j+} or θ_1^{j-}). While considering each value of $j \in [1, n]$, one can define in this manner, piecewise, a continuous function $\theta(k)$ running throughout the complete consumer space.[9] This function is illustrated in Figure 2 in the environment of bus operator j. Every consumer who is situated above of the $\theta(k)$ function, purchases from the railway company. Consumers underneath $\theta(k)$ buy from their preferred bus operator. The distinct market areas can be identified by the different shadings.

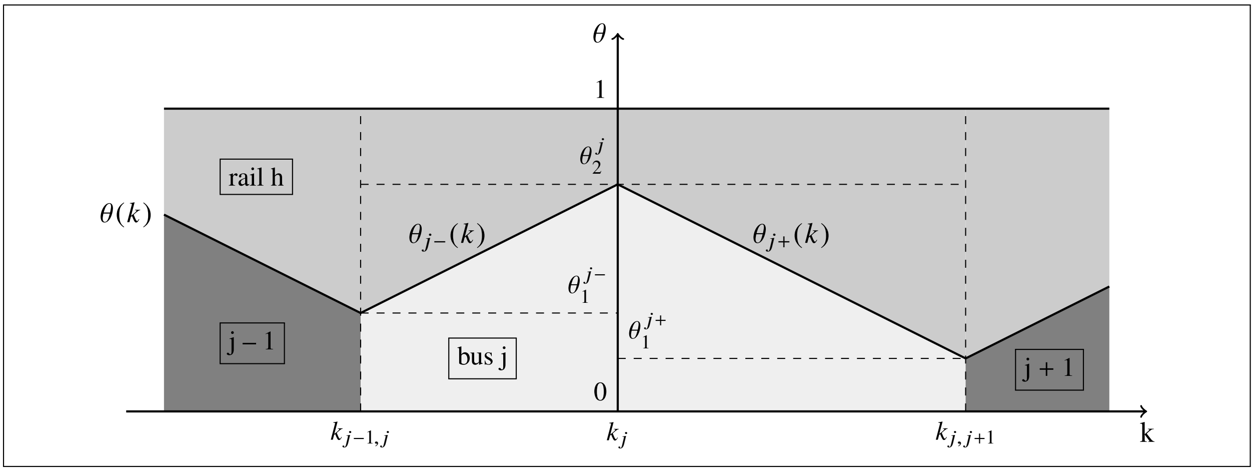

Figure 2: Market segments in the two-dimensional consumer space

Following the above considerations, the demand for the product of bus operator j corresponds to the area underneath the function $\theta(k)$ in the interval $k_{j-1,j}$ to $k_{j,j+1}$, which is shown as the bright area in figure 2. Demand is expressed as a function of prices p.[10] Please note that $\theta(k)$ and the two boundaries themselves contain the prices as arguments. Since the mass of consumers is equal to one, the railway company's demand can be expressed as the difference between one and the sum of all bus companies' demands. Hence, given our preliminary assumptions that $\theta_1^{j+}, \theta_1^{j-} > 0$ and $\theta_2^j < 1$ for all j, we obtain the following demand expressions:

$$D_j(p) = \int_{k_{j-1,j}}^{k_{j,j+1}} \theta(k)dk \;\; = \int_{k_{j-1,j}}^{k_j} \theta_{j-}(k)dk + \int_{k_j}^{k_{j,j+1}} \theta_{j+}(k)dk \qquad \text{(for every j)}$$

$$D_h(p) = 1 - \sum_{j=1}^{n} D_j(p)$$

The demand function of a bus operator can be simplified by forming the anti-derivatives and plugging in the upper and lower boundaries. Furthermore, $k_{j-1,j}$ and $k_{j,j+1}$ can be substituted by applying expression (1) and $k_j = \frac{j-1}{n}$, respectively. Thereafter, $D_h(p)$ can also be simplified. Because of the circular shape of the market, we have $p_{n+1} = p_1$ and $p_{1-1} = p_n$, which allows us to abbreviate the expression within the sum.[11]

$$\begin{aligned} D_j(p) = \frac{1}{t\,(q_h - q_l)} & \left[\frac{1}{2} \left(\left(p_{j-1} + p_{j+1}\right)(p_h + tR) - \frac{{p_{j-1}}^2 + {p_{j+1}}^2}{4} \right) \right. \\ & \left. - p_j \left(\frac{p_{j-1} + p_{j+1}}{4} - \frac{3}{4} p_j + p_h + tR + \frac{t}{2n} \right) + \frac{t}{n} \left(p_h + tR - \frac{t}{4n} - \frac{p_{j-1} + p_{j+1}}{4} \right) \right] \end{aligned} \tag{2}$$

$$D_h(p) = 1 - \frac{1}{t\,(q_h - q_l)} \left[t \left(p_h + tR - \frac{t}{4n} \right) + \sum_{j=1}^{n} \left(\frac{1}{2} {p_j}^2 - p_j \left(\frac{p_{j-1}}{2} + \frac{t}{n} \right) \right) \right] \tag{3}$$

[9] The function is continuous at $k_{j,j+1}$ as well, since consumers of types $(k_{j,j+1}, \theta_1^{j+})$ are indifferent between the operators h, j and $j+1$. Similarly, the function is continuous at $k_{j,j-1}$ as well. In other words, it holds $\theta_1^{j+} = \theta_1^{(j+1)-}$ and $\theta_1^{j-} = \theta_1^{(j-1)+}$. Because the consumer space is actually a cylinder, $\theta(k)$ starts and ends at the same point.

[10] We use the price vector $p = (p_h, p_1, \ldots, p_n)$ although the demand of one bus operator must not necessarily depend directly on the prices of every other bus operator in the market.

[11] Whereas the calculation of $D_j(p)$ can be easily reconstructed using a computer algebra system, the simplification of $D_h(p)$ is not as obvious. Hence, in Appendix A.1 the applied manipulations are shown.

In this paper, we focus on the case that $\theta_1^{j+}, \theta_1^{j-} > 0$ and $\theta_2^j < 1$ for all j. However, it is quite clear from the above, how the set of consumers taking bus j resp. the railway will look like if these conditions are not met: If $\theta_1^{j-} < 0$ then the $\theta_{j-}(k)$-curve will cut the 0-axis (compare Figure 2), and the set of consumers taking bus j is given by the area between the 0-axis and $\max\{0, \theta_{j-}(k)\}$, whereas the consumers above $\max\{0, \theta_{j-}(k)\}$ will take the rail. Similarly for the case that $\theta_1^{j+} < 0$. Finally, if $\theta_2^j > 1$, then the $\theta_{j-}(k)$-curve will cut the constant 1-curve, and the set of consumers taking the railways is given by the area between 1 and $\min\{1, \theta_{j-}(k)\}$, while the consumers below $\min\{1, \theta_{j-}(k)\}$ will take bus j. The corresponding demand expressions are given in Appendix A.3.

3.2 Price competition

We assume that all suppliers in the market face the same constant costs of production, which therefore can be normalized to zero. Referring to (2) and (3), the profits in this model are: $\pi_j(p) = p_j D_j(p)$ for all j and $\pi_h(p) = p_h D_h(p)$. Thus, the first-order conditions that must necessarily hold in a price equilibrium are as follows.

$$\frac{\partial \pi_j(p)}{\partial p_j} = D_j(p) - \frac{p_j}{t\,(q_h - q_l)} \left(p_h + tR - \frac{3}{2} p_j + \frac{p_{j-1} + p_{j+1}}{4} + \frac{t}{2n} \right) \stackrel{!}{=} 0 \quad \text{(for every j)}$$

$$\frac{\partial \pi_h(p)}{\partial p_h} = D_h(p) - \frac{p_h}{q_h - q_l} \stackrel{!}{=} 0$$

Without further considerations, it is difficult to say how many (if any) solutions this system of equations has. Because of the symmetrical character of the model, we expect to find one solution where all bus operators choose to set the same price, denoted by p_l. In this 'quasi–symmetric' case, the above $n + 1$ first-order conditions for an equilibrium can be reduced down to two by substituting $p_j = p_l$ for all j.

$$n p_l \,(tR + p_h - p_l) - t \left(p_h - \frac{3}{2} p_l + tR - \frac{t}{4n} \right) = 0 \tag{4}$$

$$p_h = \frac{1}{2} \left(q_h - q_l - tR + p_l + \frac{t}{4n} \right) \tag{5}$$

The two equations (4) and (5) are linear in p_h, but quadratic in p_l. Hence, there are two distinct pairs of prices that solve these equations. Appendix A.2 shows that only one of these pairs may constitute a Nash equilibrium. Introducing the notation:

$$A := q_h - q_l + tR \quad \text{and} \quad B := 16A^2 + 8A\frac{t}{n} + 97\left(\frac{t}{n}\right)^2 \tag{6}$$

(implying $A > 0$ and $B > 0$) the candidate price equilibrium is:

$$p_l^N = \frac{1}{2} \left(A + \frac{9}{4} \frac{t}{n} \right) - \frac{1}{8} \sqrt{B} \tag{7}$$

$$p_h^N = \frac{1}{4} \left(3A - 4tR + \frac{11}{4} \frac{t}{n} \right) - \frac{1}{16} \sqrt{B} \tag{8}$$

If we substitute these prices into the original demand functions, we get the respective quantities sold in equilibrium. These are denoted by x_h^N for the railway company and for every bus operator by x_l^N. After some rearrangements one obtains:

$$x_l^N = \frac{1}{4n}(1 - x_h^N)$$

$$x_h^N = \frac{1}{q_h - q_l} p_h^N$$

Note that, in a quasi–symmetric equilibrium, expression (1) reduces to $k_{j,j+1} = \frac{2j-1}{2n}$, where $k_j = \frac{j-1}{n}$ has been used. This in turn implies that the critical values θ_1^{j+} and θ_1^{j-} become identical for any given j and also among the different j. Defining $\theta_1 := \theta_1^{j+} = \theta_1^{j-} \, \forall j$, in a quasi–symmetric equilibrium it holds $\theta_1 = \frac{p_h^N - p_j^N + tR - \frac{t}{2n}}{q_h - q_l}$. Likewise, the θ_2^j are in a quasi–symmetric equilibrium the same for all j, and introducing similar notation it holds $\theta_2 = \frac{p_h^N - p_j^N + tR}{q_h - q_l}$.

While deriving these equilibrium results, it was assumed that all θ_1^{j+}, θ_1^{j-} and θ_2^j are greater than zero, respectively less than one. We can now state parametric conditions such that in the candidate equilibrium the values of θ_1 and θ_2 satisfy these conditions:[12]

$$\theta_1 > 0 \quad \Leftrightarrow \quad p_h^N - p_l^N > \frac{t}{2n} - tR \quad \Leftrightarrow \quad A = q_h - q_l + tR > \frac{t}{n} \tag{9}$$

$$\begin{aligned} \theta_2 < 1 \quad &\Leftrightarrow \quad p_h^N - p_l^N < q_h - q_l - tR \\ &\Leftrightarrow \quad 3\frac{t^2}{n^2} + 4\frac{t^2}{n}R > 8(q_h - q_l)^2 + (q_h - q_l)\left(10\frac{t}{n} - 8tR\right) \end{aligned} \tag{10}$$

Appendix A.3 shows that, given these two conditions, the proposed equilibrium is indeed a Nash equilibrium, i.e. that no player has a strict incentive to deviate from. There, we also consider deviations that lead to violations of the standard case with $\theta_1^{j+}, \theta_1^{j-} > 0$ and $\theta_2^j < 1$ for all j.

3.3 Market entry

In the last section, the second stage of the two stage game was analysed. In this price competition, the number of market participants was regarded as fixed. Now, in the first stage of this model, the focus is on the number of bus operators that will enter the market. Remember the railway company always occupies the circle center. Making their entry decision, the bus operators anticipate the outcome of the second stage, thus the price equilibrium. So, the profit function in the market entry stage (without entry costs) which is the same for all bus operators and dependent on their number n is described using the expressions derived before.

$$\pi_l(n) = p_l^N(n) \cdot x_l^N(n)$$

According to the assumed zero-profit condition, bus operators will enter the market as long as their expected profits cover at least the entry costs F. Hence, in the Nash-equilibrium of the first stage, the number of bus operators must satisfy the condition: $\pi_l(n^N) = F$.[13] Unfortunately, this equation cannot easily be solved for n^N as the profit function contains n among others to the minus third power. This also means, there may exist multiple candidates for a solution. Furthermore, it must be assured that the profit function is decreasing in the relevant range of n, so that an increase in market participants reduces profits of the bus operators. Due to these difficulties to present a general solution in the market entry stage, we content ourselves with demonstrating the underlying idea with the aid of a numerical example.

The upper section of Table 1 shows the parameters used in this example. Regarding the product qualities in this model, only the difference between q_h and q_l is relevant, which is set equal to one. Also t is equal to one, whereas R is set to $\frac{1}{12}$. Before the equilibrium values of the prices, quantities and so on can be calculated, at first n^N has to be determined. When substituting said parameters into the profit function of the first stage, we obtain:[14]

$$\pi_l(n) = p_l^N(n) \cdot x_l^N(n) = \frac{1}{1152n^3}\left(60\sqrt{169n^2 + 78n + 873} - 156n - 1764\right)$$

[12]Use $p_h^N - p_l^N = \frac{1}{4}A - tR - \frac{7}{16}\frac{t}{n} + \frac{1}{16}\sqrt{B}$. For expression (9) the initial inequality is equivalent to $B > (15t/n - 4A)^2$ and this is equivalent to $A > t/n$.

[13]We ignore the integer constraint on n. It could easily be re-introduced by use of the floor function.

[14]Please note that the profit function was slightly simplified using that n is assumed to be strictly positive.

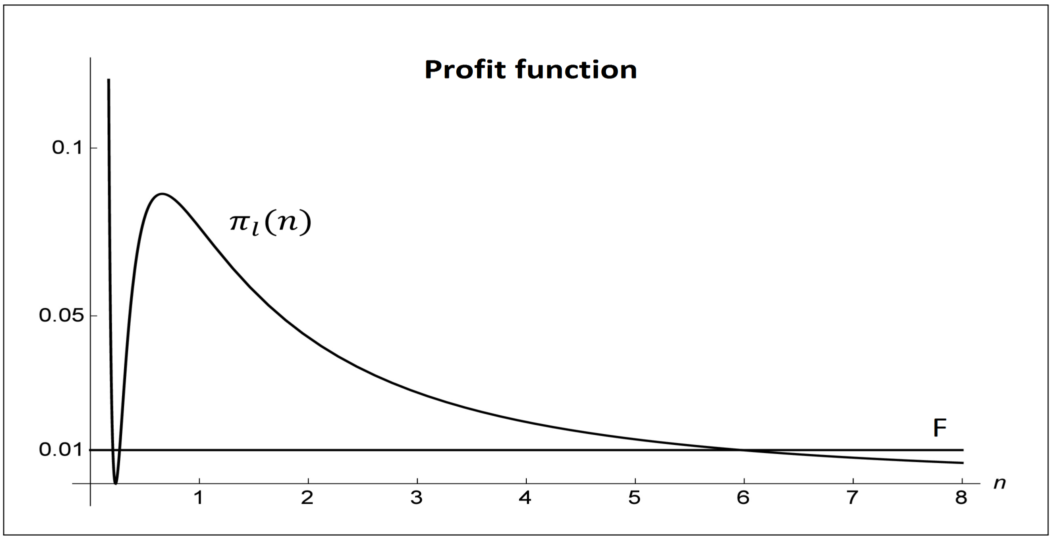

Figure 3: Profit function

The behavior of this function is depicted in Figure 3 which also shows the market entry costs F chosen to be 0.01. There are three intersections of the profit function with the horizontal entry costs F. Two of the corresponding values of n, however, are less than one, namely $n_1 \approx 0.21$ and $n_2 \approx 0.27$. This means, they do not need to be considered as equilibria, since n should be at least as large as 1 (in fact, we assumed n to be bigger than two). Hence, the only useful candidate in this case is $n_3 \approx 5.97$. We furthermore see that the profit function is decreasing in this point, i.e. additional entry would lead to losses. Therefore $n^N \approx 5.97$ constitutes the equilibrium on the model's entry stage in this numerical example.

Using this result, the equilibrium values of the prices, quantities and thereby profits can be calculated. The outcomes are shown in the lower section of Table 1 (rounded to two decimal places). The railway company's price is about four times the price of a bus. Its market share is about 54%, whereas the bus operators divide the rest among them. The latter's profit is equal to F due to the after-entry zero-profit condition. Also the values of θ_1 and θ_2 are determined. For 9% of the consumers (the difference $\theta_2 - \theta_1$), their position on the circle has an impact on their decision, given the prices in this example. These consumers choose to equal parts the railway company or one of the bus operators. This is already apparent from Figure 2 and due to the piecewise linear shape of the function $\theta(k)$. Altering the way the horizontal distance is assessed in the utility function of the model could however lead to different outcomes. At last, the conditions established at the end of the previous section can be reviewed. These are satisfied, since we have $\theta_1^N > 0$ and $\theta_2^N < 1$. Thus, there exists a non-empty set of parameter combinations for this model to produce valid results.

Parameters	$q_l = 0$	$q_h = 1$	
	$t = 1$	$R = 0.12$	$F = 0.01$
Equilibrium	$n^N = 5.97$	$\theta_1^N = 0.41$	$\theta_2^N = 0.50$
	$p_h^N = 0.54$	$x_h^N = 0.54$	$\pi_h^N = 0.30$
	$p_l^N = 0.13$	$x_l^N = 0.08$	$\pi_l^N = 0.01$

Table 1: Numerical example for the general model

4 Simplified model

In the above model, we were not able to determine an explicit solution for the first stage equilibrium n^N which then could be substituted into the second stage equilibrium. But this would be necessary, e.g., in order to examine the effect of parameter changes on the market equilibrium using comparative statics. Therefore, we now consider a slightly simplified version of the model. So far, the consumers' preferences for quality θ were continuously and uniformly distributed between zero and one. It was shown that, under certain conditions, in the market equilibrium, the consumers fall into three categories according to their respective quality θ. Consumers with a high value of θ always prefer the railway over the bus, independent of their position on the circle k_i. On the opposite side, consumers with a low preference for quality, always choose one of the bus operators, depending on their position on the circle. Only consumers with intermediate θ make an intermodal choice between bus and rail that depends on their exact combination of (k_i,θ_i). These three categories and their boundaries (θ_1^N and θ_2^N) are dependent on the equilibrium prices and thus generated within the model. It is clear that there cannot be another basic category of consumers in this model. So the three categories are in a sense generic to our model with two-dimensionally distributed consumers.

In the simplified model, we seek to externalise these generic categories. Instead of using a continuous preference for quality, we introduce three distinct levels θ_z for $z \in \{1,2,3\}$, where $0 \leq \theta_1 < \theta_2 < \theta_3 \leq 1$. We denote by λ_z the number of consumers of level θ_z, with $0 < \lambda_z < 1$ and $\sum_z \lambda_z = 1$. Consumers of every θ-level are evenly distributed around the circle. Thus, on every section of the circle with length r there are $r\lambda_z$ consumers of level θ_z. We assume that all consumers of levels 2 and 3 prefer the railways over buses if prices are the same: $tR < \theta_2(q_h - q_l)$.

Below (in section 4.3), we will state appropriate assumptions on parameters which imply that, in the market equilibrium, these three θ-levels represent the three generic categories of consumers that have been identified in the general model. That is, consumers of level θ_3 will take the rail, consumers of level θ_2 will choose between a bus and the rail, and consumers of level θ_1 will choose between buses only. The virtue of the simplification is, of course, that we do not have to track changes of the critical θ-boundaries between the generic categories (the θ_1^N and θ_2^N of the general model) any more when analysing equilibria or comparative statics.

Since, in the general model, θ is uniformly distributed, one might set $\lambda_z = 1/3$ all z. But this assumption is not imposed, so that we introduce non-uniform distributions of θ in the simplified model. However, in the spirit of this model, it is senseful to set $\lambda_2 \geq 1/3$ since, for one thing, level 2 is of particular interest to us and, also, this avoids two-peak distributions. As this latter assumption will be needed for some specific comparative statics results only, it is not generally imposed in the following, but will explicitly be mentioned when needed.

4.1 Demand in the simplified model

In this version of the model, competition occurs in two different ways. Firstly, a bus operator competes with the bus operator in front and behind it on the circle for consumers with θ_1. We already know that consumers between $k_{j-1,j}$ and $k_{j,j+1}$, as calculated in (1), prefer bus j over every other bus, irrespectively of the exact value of θ_1. In Figure 4a, the solid line indicates the consumers with θ_1 who choose bus j in a market with four bus operators.

Secondly, there is competition between the railway company and each bus operator for consumers with θ_2. We will set parameters such that, for every j, the consumer $(k_{j,j+1},\theta_2)$ chooses the railway company, whereas consumer (k_j,θ_2) prefers bus j in equilibrium. Consumers between k_j and $k_{j,j+1}$ have to decide between bus j and rail h, and there exists a consumer denoted by (k_{j+},θ_2), who is indifferent between these two options. The position k_{j+} of the indifferent consumer is strictly between k_j and $k_{j,j+1}$ and can be determined as follows.

$$\begin{aligned} & V_0 + \theta_2 q_l - p_j - t(k_{j+} - k_j) = V_0 + \theta_2 q_h - p_h - tR \\ \Leftrightarrow \quad & k_{j+} = \frac{1}{t}\Big(p_h - p_j + tR - \theta_2(q_h - q_l)\Big) + k_j \end{aligned} \tag{11}$$

Analogously, k_{j-} can be calculated, which is the position of the consumer with θ_2 who also is indifferent between bus j and rail h but situated between $k_{j-1,j}$ and k_j. Figure 4b shows the segmentation of the market in the environment of bus operator j for consumers with θ_2. Again, consumers between $k_{j-1,j}$ and $k_{j,j+1}$ prefer bus j over every other bus. But now, consumers on the dashed line prefer the railway company over bus j, whereas consumers closer to k_j still choose bus j.

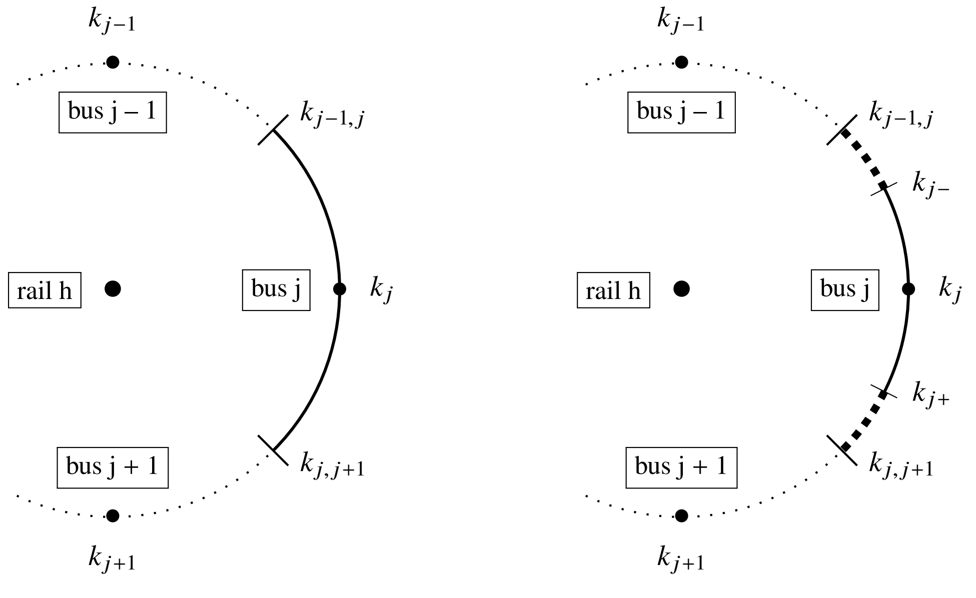

Figure 4: Segmentation of a market with four bus operators in the environment of bus j in the simplified model

As just explained, the demand for bus operator j consists of two parts: Firstly, every consumer with θ_1 between $k_{j-1,j}$ and $k_{j,j+1}$, and secondly, the consumers with θ_2 between k_{j-} and k_{j+}. Also the railway companies' demand consists of two parts, namely every consumer with θ_3 and among the consumers with θ_2 those between $k_{j-1,j}$ and k_{j-} as well as between k_{j+} and $k_{j,j+1}$, for every j. Hence, the respective demand functions can be expressed as:

$$\begin{aligned} D_j(p) &= \lambda_1(k_{j,j+1} - k_{j-1,j}) + \lambda_2(k_{j+} - k_{j-}) \qquad \text{(for every j)} \\ D_h(p) &= \lambda_3 + \lambda_2 \sum_{j=1}^{n} \Big[(k_{j-} - k_{j-1,j}) + (k_{j,j+1} - k_{j+})\Big] \end{aligned}$$

In these demand functions, the positions of the indifferent consumers, as calculated in (1) and (11), can be substituted. Furthermore, using $k_j = \frac{j-1}{n}$ and $\lambda_3 = 1 - \lambda_2 - \lambda_1$, we can simplify and rearrange. As applied in appendix A.1, in the railway company's demand function, the expression $\sum_{j=1}^{n}(p_{j-1} + p_{j+1})$ can be replaced by $\sum_{j=1}^{n} 2p_j$. This gives:

$$\begin{aligned} D_j(p) &= \frac{\lambda_1}{t}\left(\frac{p_{j-1} + p_{j+1}}{2} - p_j + \frac{t}{n}\right) + \frac{2\lambda_2}{t}\Big(p_h - p_j + tR - \theta_2\left(q_h - q_l\right)\Big) \qquad \text{(for every j)} \\ D_h(p) &= 1 - \lambda_1 - \frac{2\lambda_2 n}{t}\left(p_h + tR - \theta_2(q_h - q_l)\right) + \frac{2\lambda_2}{t}\sum_{j=1}^{n} p_j \end{aligned}$$

4.2 Price competition

Again, we start by examining the second stage of the two-stage game. At this stage, the relevant profit functions are $\pi_h(p) = p_h D_h(p)$ and, for every j, $\pi_j(p) = p_j D_j(p)$. Thus, the first-order conditions are as follows.

$$\frac{\partial \pi_h(p)}{\partial p_h} = D_h(p) - \frac{2\lambda_2 n}{t} p_h \stackrel{!}{=} 0 \qquad \text{and} \qquad \frac{\partial \pi_j(p)}{\partial p_j} = D_j(p) - \frac{\lambda_1 + 2\lambda_2}{t} p_j \stackrel{!}{=} 0 \quad (\forall j)$$

Note that the second derivatives of the profit functions are negative, hence the prices characterized by the first-order conditions refer to unique profit maxima. From the first-order conditions, we get the following reaction functions.

$$p_h(p) = \frac{t(1-\lambda_1)}{4\lambda_2 n} - \frac{1}{2}\left(tR - \theta_2(q_h - q_l)\right) + \frac{1}{2n}\sum_{j=1}^{n} p_j$$

$$p_j(p) = \frac{1}{\lambda_1 + 2\lambda_2}\left[\frac{\lambda_1}{2}\left(\frac{p_{j-1} + p_{j+1}}{2} + \frac{t}{n}\right) + \lambda_2\left(p_h + tR - \theta_2\left(q_h - q_l\right)\right)\right] \quad (\forall j)$$

Summing up $p_j(p)$ for all j, we can express the sum over all p_j, which only depends on p_h.

$$\sum_{j=1}^{n} p_j = \frac{2n\lambda_2\left(p_h + tR - \theta_2(q_h - q_l)\right) + t\lambda_1}{\lambda_1 + 4\lambda_2}$$

Substituting this into the reaction function $p_h(p)$ already yields a unique solution for p_h, the price of the railway company. In addition, we obtain a system of n linear equations $p_j(p)$ and n variables, namely, all bus prices $p_1, \ldots, p_n$. Considering the symmetry of the model, we find a solution with identical bus prices $p_j = p_l$ for every j. Hence, there is a Nash-equilibrium candidate of the second stage of the game with the following prices.

$$p_h^N = \frac{1}{2(3\lambda_2 + \lambda_1)}\left((\lambda_1 + 2\lambda_2)\left(\theta_2(q_h - q_l) - tR - \frac{\lambda_1 - 1}{\lambda_2}\frac{t}{2n}\right) + \frac{t}{n}\right)$$

$$p_l^N = \frac{1}{3\lambda_2 + \lambda_1}\left(\lambda_2\left(tR - \theta_2(q_h - q_l)\right) + (\lambda_1 + 1)\frac{t}{2n}\right)$$

The corresponding equilibrium quantities can be determined from the demand functions.

$$x_h^N = \frac{n\lambda_2}{(3\lambda_2 + \lambda_1)t}\left((\lambda_1 + 2\lambda_2)\left(\theta_2(q_h - q_l) - tR - \frac{\lambda_1 - 1}{\lambda_2}\frac{t}{2n}\right) + \frac{t}{n}\right)$$

$$x_l^N = \frac{\lambda_2(\lambda_1 + 2\lambda_2)}{(3\lambda_2 + \lambda_1)t}\left(tR - \theta_2(q_h - q_l) + \frac{\lambda_1 + 1}{\lambda_2}\frac{t}{2n}\right)$$

It seems worth noting that the characterization of this candidate equilibrium does not contain the parameters θ_1 or θ_3, whereas θ_2 appears in a simple way. However, we have to make sure that the generic constellation of demand is satisfied in this equilibrium.

4.3 Conditions for the relevant type of equilibrium

Before turning to the first stage of the game, we want to point out and review the assumptions used in the above analyses of demand and price equilibria. At first, we assumed that every consumer with θ_3 prefers the railway company. This means that, even if the distance on the circle to a bus is equal to zero, the utility from choosing the train would exceed the utility from taking this bus.

$$V_0 + \theta_3 q_h - p_h^N - tR > V_0 + \theta_3 q_l - p_l^N \qquad \Leftrightarrow \qquad \theta_3(q_h - q_l) - tR > p_h^N - p_l^N \tag{12}$$

Secondly, we have assumed that consumers with θ_1 always choose one of the buses. Since in the candidate equilibrium all bus prices are equal, $k_{j,j+1}$ is the same for all j and is equal to $k_j + \frac{1}{2n}$. It is sufficient to claim that consumer $(k_{j,j+1}, \theta_1)$ prefers bus j over the railways.

$$V_0 + \theta_1 q_l - p_l^N - \frac{t}{2n} > V_0 + \theta_1 q_h - p_h^N - tR \qquad \Leftrightarrow \qquad \theta_1(q_h - q_l) - tR < p_h^N - p_l^N - \frac{t}{2n} \tag{13}$$

In addition there are two more conditions that must hold in the equilibrium in order to get the generic constellation of demand. They have to assure that some customers with θ_2 take the railways and some of them choose a bus, so that the market is divided as depicted in Figure 4, with $k_{j-1,j} < k_{j-} < k_j < k_{j+} < k_{j,j+1}$.

$$V_0 + \theta_2 q_l - p_l^N > V_0 + \theta_2 q_h - p_h^N - tR \qquad \Leftrightarrow \qquad \theta_2(q_h - q_l) - tR < p_h^N - p_l^N \tag{14}$$

$$V_0 + \theta_2 q_h - p_h^N - tR > V_0 + \theta_2 q_l - p_l^N - \frac{t}{2n} \qquad \Leftrightarrow \qquad \theta_2(q_h - q_l) - tR > p_h^N - p_l^N - \frac{t}{2n} \tag{15}$$

By plugging in the price expressions of the candidate equilibrium, these conditions could be re-stated as conditions on model parameters. However, since the resulting expressions do not lend themselves to useful simplifications (the number of parameters in the simplified model is quite large), we omit stating them here. It should be noted that condition (14) implies an upper bound and (12) a lower bound for n.[15]

Further note that the conditions stated here are only necessary, not sufficient for guaranteeing that the equilibrium candidate derived in the last section is in fact a Nash equilibrium of the price game. As in the general model, showing a Nash equilibrium would require to look at pricing behavior of the individual players (the rail and a bus operator) that might violate the generic constellation of demand even if all other players play the equilibrium strategies. Deriving sufficient conditions for the Nash equilibrium in a general way is not useful in the simplified model, due to the large number of parameters. Instead, we explain in Appendix A.4 how any numerical example can be checked in a math program whether it generates our equilibrium candidate as Nash equilibrium. Moreover, we give a numerical example that does so (thus proving that our conditions do not refer to an empty set), and for this example we also investigate the range of variations of the variable n such that the example remains valid.

4.4 Market entry

In the first stage of the game, a bus operator enters the market as long as it expects profits to at least cover the entry costs F. The profit function of a bus operator, before deducing F, depends on the number of buses in the market and is denoted by $\pi_l(n)$. It is determined by multiplying the bus price and quantity from the second stage equilibrium.

$$\pi_l(n) = p_l^N \cdot x_l^N = \frac{\lambda_1 + 2\lambda_2}{t} \left(\frac{-\lambda_2 \left(\theta_2(q_h - q_l) - tR\right) + \frac{t}{2n}(\lambda_1 + 1)}{3\lambda_2 + \lambda_1} \right)^2 \tag{16}$$

Appendix A.5 shows that the expression which is taken to the square is already positive as long as condition (14) holds. It follows that $\pi_l(n)$ is strictly decreasing in n. There is a number of buses n for which profit would become zero (as $tR - \theta_2(q_h - q_l) < 0$), however, this implies that condition (14) is already violated by a smaller value of n. For n close to zero $\pi_l(n)$ becomes arbitrarily large. Of course, the profit expression (16) is only valid as long as the equilibrium conditions (12) to (15) and $n \geq 2$ are satisfied.

The equilibrium number of firms in the market is uniquely determined by the condition $\pi_l(n^N) = F$ (ignoring integer constraint on n). Please note that due to the square in the profit function (16), there are up to

[15] This can be shown using the assumption $\theta_2(q_h - q_l) - tR > 0$. The conditions also imply concavity within the generic constellation. Parameters have to be chosen such that the lower bound is below the upper bound for n. But even stricter conditions are implied in the following.

two solutions. But due to the monotonicity, only the following solution has a positive value, which is the Nash-equilibrium of the first stage of the game.

$$n^N = \frac{t}{2}(\lambda_1 + 1)\left(-\lambda_2\left(tR - \theta_2(q_h - q_l)\right) + \sqrt{\frac{Ft}{\lambda_1 + 2\lambda_2}}(3\lambda_2 + \lambda_1)\right)^{-1} \tag{17}$$

For any numerical example, the choice of F has to be confined to an interval such that n^N is at least 2 and conditions (12) to (15) are met.[16]

4.5 Comparative statics

In this section, we examine the impact of parameter changes on the market's (subgame perfect) Nash-equilibrium. This means, we analyse how equilibrium prices, quantities, and profits as well as the number of market participants react to a marginal variation of one of the model's parameters, whilst the others are fixed. For this purpose, we substitute n^N into the outcomes of the second market stage and differentiate the resulting expressions with respect to the parameter considered as variable.

As an example, let us take a look at the railway company's price when performing comparative statics with respect to its quality q_h.

$$\frac{\partial\, p_h^N(q_h, n^N(q_h))}{\partial\, q_h} = \frac{\theta_2}{\lambda_1 + 1} > 0$$

The sign of this derivative is always positive. Hence, an increase in the railway company's product quality leads ceteris paribus to a higher equilibrium price of the railways.

The results of the comparative statics are displayed in Tables 2 and 3. In each table, the left column shows the respective parameter which is considered as variable, where the arrow shows the direction of movement. The other columns show the impacts on the various outcomes of the model. An upward arrow indicates a positive value of the derivative and therefore an increase in the viewed outcome, vice versa for a downward arrow. The equality sign is used if the derivative is equal to zero, i.e. if the respective outcome is not dependent on the considered parameter.

Parameter	**Impact on equilibrium value**						
$q_h \uparrow$	$n \downarrow$	$p_h \uparrow$	$x_h \uparrow$	$\pi_h \uparrow$	$p_l =$	$x_l =$	$\pi_l =$
$q_l \uparrow$	$n \uparrow$	$p_h \downarrow$	$x_h \downarrow$	$\pi_h \downarrow$	$p_l =$	$x_l =$	$\pi_l =$
$t \uparrow$	$n \uparrow$	$p_h \updownarrow$	$x_h \downarrow$	$\pi_h \updownarrow$	$p_l \uparrow$	$x_l \downarrow$	$\pi_l =$
$t \uparrow$ with tR constant	$n \uparrow$	$p_h \uparrow$	$x_h \downarrow$	$\pi_h \uparrow$	$p_l \uparrow$	$x_l \downarrow$	$\pi_l =$
$R \uparrow$	$n \uparrow$	$p_h \downarrow$	$x_h \downarrow$	$\pi_h \downarrow$	$p_l =$	$x_l =$	$\pi_l =$
$F \uparrow$	$n \downarrow$	$p_h \uparrow$	$x_h \downarrow$	$\pi_h \uparrow$	$p_l \uparrow$	$x_l \uparrow$	$\pi_l \uparrow$

Table 2: Comparative statics in the simplified model: parameters on qualities and costs

The column most easily understood is that of π_l, since $\pi_l = F$ by construction. Many other derivatives are also easily derived and most results are quite intuitive. For the parameter t, we were able to show

[16] In Appendix A.4, we give a numerical example satisfying the Nash equilibrium with the generic constellation of demand. For that example, we also give the bounds for possible variations of n. It ranges from $n = 4$ to $n = 8$. This translates into an interval for F ranging from $F = 0.0015$ to $F = 0.0077$.

by numerical examples that p_h and π_h can move in different directions, depending on the other model parameters. Hence, double-headed arrows were assigned in Table 2. We also added a row, showing the comparative statics of an increase of t while R is simultaneously decreased, such that the product tR is held constant. In other words, transport costs are increased on the circumference, but not to the center of the circle. For this case, the table shows that buses respond by raising their prices, as competition among them becomes weaker, and at the same time more buses are entering the market. The railway operator also increases its price. The individual quantities of buses x_l are decreasing; however, the market share of all buses nx_l is increasing, as is implied by the decrease in railway's quantity x_h.[17]

If buses' costs F are increased, some buses will have to leave the market, allowing the others to increase their prices. The train operator will then also increase its price. These three initial movements have different effects on each buses' individual quantity x_l (it increases in response to a lower n and to a higher p_h, but decreases in response to a higher p_l). The overall effect on x_l is positive. The three initial movements have also different effects on train operator's quantity x_h (it increases in response to a lower n and to higher p_l, but decreases in response to a higher p_h). Somewhat surprisingly, the overall effect on x_h is negative, which implies that the market share of all buses nx_l is increasing in response to an increase of F. However, there is a lower bound for railways' market share x_h which is well above λ_3.

Parameter	**Impact on equilibrium value**						
$\theta_1 \uparrow$	$n =$	$p_h =$	$x_h =$	$\pi_h =$	$p_l =$	$x_l =$	$\pi_l =$
$\theta_2 \uparrow$	$n \downarrow$	$p_h \uparrow$	$x_h \uparrow$	$\pi_h \uparrow$	$p_l =$	$x_l =$	$\pi_l =$
$\theta_3 \uparrow$	$n =$	$p_h =$	$x_h =$	$\pi_h =$	$p_l =$	$x_l =$	$\pi_l =$
$\lambda_1 \uparrow \lambda_2 \downarrow$	$n \uparrow$	$p_h \downarrow$	$x_h \downarrow$	$\pi_h \downarrow$	$p_l \uparrow$	$x_l \downarrow$	$\pi_l =$
$\lambda_1 \uparrow \lambda_3 \downarrow$	$n \uparrow^*$	$p_h \downarrow$	$x_h \downarrow$	$\pi_h \downarrow$	$p_l \downarrow$	$x_l \uparrow$	$\pi_l =$
$\lambda_2 \uparrow \lambda_3 \downarrow$	$n \downarrow$	$p_h \downarrow$	$x_h \uparrow$	$\pi_h (\downarrow)$	$p_l \downarrow$	$x_l \uparrow$	$\pi_l =$

*The assumption $\lambda_2 > \frac{1}{3}$ has been used.

Table 3: Comparative statics in the simplified model: parameters of the distribution of θ

Table 3 shows the comparative statics when parameters of the distribution of θ are changed, where a change of a probability λ_z is always accompanied by an opposite change of another probability. Marginal increases of either the lowest quality level θ_1 or the highest quality level θ_3 have no impact at all, since the desirability of quality affects the intermodal competition between buses and trains, which takes place on level θ_2 only. An increase of that level gives railways an advantage with respect to the buses and leads to a decrease of the number of buses, while the behavior of the buses is otherwise unchanged.

If the share of customers on the lowest level λ_1 is increased at the cost of another level's share, railway's market share will decrease and more buses will enter the market (here is the only case where we made use of the assumption $\lambda_2 > 1/3$). If the share of customers on the intermediate level λ_2 is increased at the cost of another level's share, the number of buses will decrease and the market share of the railway increases. In particular, if λ_2 is increased at the cost of λ_3, price competition gets stronger, with all prices decreasing. In this case, it was hard to analyse the overall effect on railway's profit. Since we always found it to be decreasing in various numerical examples examined, a downward arrow in brackets was assigned in Table 3.

[17] Note that $nx_l + x_h = 1$ holds generally since markets are always fully covered.

5 Empirical hypotheses for the German land transport market

The comparative statics results of the simplified model, as summarized in the last section, can be used as basis for empirical investigations of the competition between intercity trains and buses in Germany. Such empirical studies have been carried out and published separately in Gremm (2018) and Gremm (2019) with the focus on factors that affect market entry of bus companies and price responses by the railway company. In these investigations each transport relation between one city and another one is viewed as a separate market, so that each relation constitutes a separate case of application of the theoretical model, and the empirical approach is a cross-sectional one. In the following, we derive the empirical hypotheses, implied by the model and tested by Gremm (2018, 2019) using suitable data.

5.1 Hypotheses on market entry of buses

Considering the new and growing market for intercity bus services, one question is: What factors affect market entry? In our analysis, we particularly focus on the changes of n due to changes of q_h, q_l, t, R and F.

Our comparative statics analysis shows that quality of transport seems to be a crucial factor for market entry. Table 2 displays that the number of bus operators n increases with an increasing quality of buses q_l and with a decreasing quality of intercity trains q_h. The smaller the quality difference between intercity trains and buses gets, the more bus operators enter the market. With regard to the intermodal competition, quality can be described by travel time or by types of trains (RE, IC, EC, ICE) and buses. Moreover, when there is the necessity of a transfer from one train to another, this is clearly perceived as a reduction of train quality q_h by customers. Therefore, we formulate the following first hypothesis about market entry:

R1: Bus operators enter particularly on lines where, ceteris paribus, quality differences between intercity trains and intercity buses are relatively small. In this sense, intercity bus operators tend to occupy profitable market niches.

Apart from the quality of transport services, our comparative analysis shows that the distance R of customers to the railway company has a positive impact on the number of bus entrants. This distance R can be interpreted – with a slight coarsening of the model setup – as average waiting time between the optimal departure time from the customers' points of view and the actual departure time. Average waiting time is increased when the frequency of train departures is reduced. There is a similar spatial aspect. If a city is not connected to the intercity train network, the access to this network is less easy, which may be captured as an increase of the parameter R. In view of the comparative statics results for a change of R, as given in Table 2, we state the following hypothesis:

R2: Bus operators enter particularly on lines where, ceteris paribus, intercity trains have a relatively low frequency or on lines which are not served by long-distance trains at all.

Furthermore, the comparative statics of Table 2 shows that an increase in the fixed costs of buses F leads to a lower number of bus operators entering the market. The reason is that, if fixed costs increase, each bus operator needs a larger market share in order to cover the expenses. Fixed costs of bus operators include staff costs, depreciation, distribution costs and tolls. In other words, it is almost all costs of bus operation, whereas variable costs, depending on the number of passengers, can be ignored. Hence, we can summarize:

R3: The number of bus operators on a line decreases when, ceteris paribus, buses' costs of servicing the line increase.

In this model, the size of the market (total number of customers) has been fixed to one. But this can be seen as a normalization relative to the buses' cost of operating a line, F. An increase in F can therefore be interpreted either as a cost increase or as a decrease of the size of the market. Hence, we arrive at the following hypothesis, as mirror image result to the preceding one:

R4: The number of bus operators on a line increases when, ceteris paribus, the size of the market for that line increases.

5.2 Hypotheses on price responses by the railway operator

After bus operators have entered a specific transport line, the intercity railway company will react with its price. In principle, the railway company will reduce its price if, for some reason, its competitive position with respect to the buses gets worse. In the inverse case, the railway company increases its price. The relative position of the railway company is influenced by railway properties (q_h, R) as well as by properties of bus companies (F, q_l).

According to Table 2, the quality difference between intercity trains and intercity buses has an impact on the price setting behavior of the railway company. The smaller the quality difference, the larger the number of bus operators and the lower the price set by the railway company. In particular, an increase in the quality of railway services leads to an increase in railway price. At the same time, an increase in bus quality (for example introducing entertainment services or a direct connection) leads to a lower railway price. Therefore, we formulate the following hypothesis:

R5: Increases of intercity bus quality lead to a lower railway price, whereas increases of railway quality lead to a higher railway price, ceteris paribus. The lower the quality difference between the transport modes, the lower is the price of railway services, ceteris paribus.

A reduction of railway frequency can be interpreted as an increase of R, as has been argued above. As implied by Table 2, this loss of quality advantage will be compensated by a decrease of the railway price. Therefore, we can formulate the following hypothesis:

R6: A lower railway frequency is accompanied by a lower railway price.

6 Conclusion

This paper analyses the intermodal competition between intercity buses and trains based on the following stylized facts of the market in Germany. The railway system is already in existence (incumbent) and intercity bus operators may enter the market. Railway transport is of higher quality than bus transport, and the network of trains is very large compared to the transport opportunities offered by buses. Bus operators are horizontally differentiated as different varieties, so that there is imperfect competition among them. Hence, we arrive at a model of a dominant high quality firm (the railway operator) vis-à-vis an oligopolistic competitive fringe of low quality firms (the bus operators). In order to model this situation, we used Economides' spatial cylinder model of vertical and horizontal product differentiation (which itself is an extension of Salop's circle model) with the addition that there is a dominant firm located in the center at the top of the cylinder, whereas the oligopolistic fringe firms are located on the circumference at the bottom of the cylinder.

In the general version of the model, consumers' tastes for quality are uniformly distributed on the unit interval. For the stage of price competition, a symmetric Nash equilbrium was derived taking account of all potential deviating strategies by the players, including potential undercutting strategies by bus operators. The market entry stage was then solved numerically. Afterwards, a simplified version of the model was introduced, with only three levels of consumers' tastes for quality, for which it was possible to carry out comparative statics analyses for the full two-stage equilibrium.

The comparative statics is needed to generate hypotheses for empirical investigation. One property shown is that the relative difference in the qualities of service between trains and intercity buses has an impact on the market entry decisions of bus companies as well as on the price setting behavior of the railway company. Thus, intercity bus operators tend to occupy niches, in which the quality advantage of railways is comparatively small. Moreover, if market size (i.e. the number of customers on a line) increases, more bus operators will enter the market and the railway company will respond by reducing its price.

The hypotheses on market entry of buses and on price reactions of the railways that were derived in this theoretical model have been tested empirically in Gremm (2018) and Gremm (2019). Both publications use the German intercity bus and railway market for their analysis. Gremm (2019) examines the market entry factors of bus operators concerning the exististing railway system. The results show that bus operators indeed enter with a higher probability and in larger numbers on those transport markets which are relatively large and on those ones where the travel time difference between buses and trains is relatively low. However, for some other quality indicators investigated, the expected effects on market entry behavior could only partially be confirmed.

The publication of Gremm (2018) addresses the relationship between the intermodal competition by buses and the pricing behavior of the railway operator DB Fernverkehr AG. It can be confirmed that the railway operator sets lower prices on larger markets and on markets with a relatively low quality difference between the railway and bus services. This applies particularly for transport markets without a direct railway connection and for markets where many IC trains, instead of the higher quality ICE trains, are operating.

In summary, our results show that the intermodal competition by intercity buses which was introduced in Germany in 2013 led to an improvement of the public intercity passenger transport system. Buses have put the railway operator under some healthy competitive pressure, which is new, since intramodal competition within the railway sector is practically inexistent in Germany so far (and airline competition is restricted to a few very long distance connections). Buses also complement the existing train services by adding services at particularly low prices and also some direct connections which are not offered by trains. The model is also able to predict the impacts of some policy measures. Currently, the introduction of a road toll for intercity buses is discussed in Germany. This would imply a mild increase in their operating costs. The model predicts that this would reduce the equilibrium number of buses entering the market and thereby weaken intermodal competition, leading to higher train fares.

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Appendix

A.1 Manipulation of the railway company's demand function in the general model

Here we derive expression (3) for the railway company's demand. In section 3.1 we stated: $D_h(p) = 1 - \sum_{j=1}^{n} D_j(p)$. In addition, due to the circular market, we know: $p_{n+1} = p_1$ and $p_{1-1} = p_n$. From this, the following simplifications can be shown to hold: $\sum_{j=1}^{n}(p_{j-1} + p_{j+1}) = \sum_{j=1}^{n} 2p_j$ and $\sum_{j=1}^{n}(p_{j-1}^2 + p_{j+1}^2) = \sum_{j=1}^{n} 2p_j^2$ and $\sum_{j=1}^{n} p_j(p_{j-1} + p_{j+1}) = \sum_{j=1}^{n} 2p_jp_{j-1}$. Using these considerations and expression (2) $D_h(p)$ can be simplified in the following manner.

$$
\begin{aligned}
D_h(p) =&1 - \sum_{j=1}^{n} \frac{1}{t\left(q_h - q_l\right)} \left[\frac{1}{2}\left(\left(p_{j-1} + p_{j+1}\right)\left(p_h + tR\right) - \frac{{p_{j-1}}^2 + {p_{j+1}}^2}{4}\right)\right. \\
&\left. -p_j\left(\frac{p_{j-1} + p_{j+1}}{4} - \frac{3}{4}p_j + p_h + tR + \frac{t}{2n}\right) + \frac{t}{n}\left(p_h + tR - \frac{t}{4n} - \frac{p_{j-1} + p_{j+1}}{4}\right)\right] \\
=&1 - \frac{1}{t\left(q_h - q_l\right)}\left[\tfrac{p_h + tR}{2}\sum_{j=1}^{n}\left(p_{j-1} + p_{j+1}\right) - \tfrac{1}{8}\sum_{j=1}^{n}\left({p_{j-1}}^2 + {p_{j+1}}^2\right) - \tfrac{1}{4}\sum_{j=1}^{n} p_j\left(p_{j-1} + p_{j+1}\right)\right. \\
&\left. + \frac{3}{4}\sum_{j=1}^{n} p_j^2 - \left(p_h + tR + \frac{t}{2n}\right)\sum_{j=1}^{n} p_j - \frac{t}{4n}\sum_{j=1}^{n}\left(p_{j-1} + p_{j+1}\right) + t\left(p_h + tR - \frac{t}{4n}\right)\right] \\
=&1 - \frac{1}{t\left(q_h - q_l\right)}\left[\frac{p_h + tR}{2}\sum_{j=1}^{n} 2p_j - \frac{1}{8}\sum_{j=1}^{n} 2p_j^2 - \frac{1}{4}\sum_{j=1}^{n} 2p_jp_{j-1}\right. \\
&\left. + \frac{3}{4}\sum_{j=1}^{n} p_j^2 - \left(p_h + tR + \frac{t}{2n}\right)\sum_{j=1}^{n} p_j - \frac{t}{4n}\sum_{j=1}^{n} 2p_j + t\left(p_h + tR - \frac{t}{4n}\right)\right] \\
=&1 - \frac{1}{t\left(q_h - q_l\right)}\left[\frac{1}{2}\sum_{j=1}^{n} p_j^2 - \sum_{j=1}^{n} p_j\frac{p_{j-1}}{2} - \frac{t}{2n}\sum_{j=1}^{n} p_j - \frac{t}{2n}\sum_{j=1}^{n} p_j + t\left(p_h + tR - \frac{t}{4n}\right)\right] \\
=&1 - \frac{1}{t\left(q_h - q_l\right)}\left[t\left(p_h + tR - \frac{t}{4n}\right) + \sum_{j=1}^{n}\left(\frac{1}{2}{p_j}^2 - p_j\left(\frac{p_{j-1}}{2} + \frac{t}{n}\right)\right)\right]
\end{aligned}
$$

A.2 Determination of the equilibrium prices in the general model

There are two pairs of prices, denoted by (p_{l1}, p_{h1}) and (p_{l2}, p_{h2}), which satisfy the two equations (4) and (5) that were derived from the first-order conditions when bus prices are assumed to be equal. Using the notation $A := q_h - q_l + Rt$ and $B := 16A^2 + 8At/n + 97(t/n)^2$, already introduced in (6), the corresponding prices are:

$$
\begin{aligned}
p_{l1} &= \frac{1}{2}\left(A + \frac{9}{4}\frac{t}{n}\right) + \frac{1}{8}\sqrt{B} \\
p_{h1} &= \frac{1}{4}\left(3A - 4tR + \frac{11}{4}\frac{t}{n}\right) + \frac{1}{16}\sqrt{B} \\
p_{l2} &= \frac{1}{2}\left(A + \frac{9}{4}\frac{t}{n}\right) - \frac{1}{8}\sqrt{B} \\
p_{h2} &= \frac{1}{4}\left(3A - 4tR + \frac{11}{4}\frac{t}{n}\right) - \frac{1}{16}\sqrt{B}
\end{aligned}
$$

The second derivative of the railway company's profit function is always negative:

$$\frac{\partial^2 \pi_h(p)}{\partial p_h^2} = -\frac{2}{q_h - q_l} < 0$$

The second derivative of the profit function of bus operator j is:

$$\frac{\partial^2 \pi_j(p)}{\partial p_j^2} = \frac{1}{2t\,(q_h - q_l)}\left(9p_j - p_{j-1} - p_{j+1} - 4p_h - 4tR - 2\frac{t}{n}\right) \tag{18}$$

If the first solution (p_{l1}, p_{h1}) is plugged into (18), i.e. p_h is substituted with p_{h1} and p_j, p_{j-1}, p_{j+1} with p_{l1}, one obtains:

$$\frac{\partial^2 \pi_j(p)}{\partial p_j^2} = \frac{1}{4t\,(q_h - q_l)}\left(A + \frac{25}{4}\frac{t}{n} + \frac{5}{4}\sqrt{B}\right)$$

This is strictly positive. Hence, (p_{l1}, p_{h1}) does not constitute a price equilibrium, since every bus operator would have a strict incentive to deviate from p_{l1}. For the second solution (p_{l2}, p_{h2}) one obtains:

$$\frac{\partial^2 \pi_j(p)}{\partial p_j^2} = \frac{1}{4t\,(q_h - q_l)}\left(A + \frac{25}{4}\frac{t}{n} - \frac{5}{4}\sqrt{B}\right)$$

This is strictly negative, as it holds

$$B > \left(\frac{4}{5}A + 5\frac{t}{n}\right)^2$$

Hence, we have shown that the second derivative of each bus operator's profit function is strictly negative for the pair of prices (p_{l2}, p_{h2}). Therefore, this pair of prices is the only candidate for a quasi–symmetric Nash equilibrium with $\theta_1 > 0$ and $\theta_2 < 1$.

A.3 Proof of the equilibrium in the general model

Throuhout section 3.1 it has been assumed that for every j it holds $0 < \min\{\theta_1^{j-}, \theta_1^{j+}\}$ and $\theta_2^j < 1$, so that, for every interval $[k_{j-1,j},\ k_j]$ and $[k_j,\ k_{j+1,j}]$, the demands for bus and railway operators are described by trapezoids as depicted in Figure 2. A quasi–symmetric equilibrium candidate has been characterized where all bus companies choose the same price $p_j = p_l^N$ and the railway company chooses the price p_h^N. These prices are given by (7) and (8). Moreover, parametric conditions (9) and (10) have been given such that the candidate equilibrium satisfies $0 < \theta_1$ and $\theta_2 < 1$.

In this appendix it will be shown that the candidate equilibrium is indeed a Nash equilibrium, i.e. that no player has a strict incentive to deviate from it. In order to do so we can focus on just one segment of the circle between $k_{j-1,j}$ and k_j for some j. If we discuss possible deviating strategies of the railway company, the bus companies are assumed to set p_l^N, so that all such segments of the circle will look alike (and the railway company's total demand and profit will be just $2n$ times the respective value on one such segment). If we discuss possible deviating strategies of one of the bus companies j, the other bus companies are assumed to set p_l^N and the railway company to set p_h^N, so that the two segments relevant for bus company j are symmetric (and its total demand and profit will be just 2 times the respective value on one segment).

However, we will have to consider the possibilities that deviating strategies imply θ_1^{j-} to be either positive or negative and θ_2^j to be either smaller or larger than 1. Thus, we address potential deviations from the equilibrium strategies in the following order:

a) the deviating rail operator sets a price such that $0 \geq \theta_1^{j-}$ and $\theta_2^j \leq 1$ are preserved (trapezoid case)
b) the rail operator sets a high price such that $\theta_2^j > 1$
c) the rail operator sets a low price such that $\theta_1^{j-} < 0$
d) the deviating bus operator j sets a price such that $0 \geq \theta_1^{j-}$ and $\theta_2^j \leq 1$ are preserved (trapezoid case)
e) bus operator j sets a high price such that $\theta_1^{j-} < 0$
f) bus operator j sets a low price such that $\theta_2^j > 1$

Note that as an extension to the last case f) we will also ask the question whether a bus operator will set its price even lower, such that his demand captures the whole of his neighbor bus's demand (the answer will be no, since the price would have to become negative even before this happens).

a) The rail operator sets a price such that $0 \geq \theta_1^{j-}$ and $\theta_2^j \leq 1$ are preserved (trapezoid case)

The trapezoid case is defined by the conditions $0 \leq \theta_1^{j-}$ and $\theta_2^j \leq 1$. Using (1) and $k_j = \frac{j-1}{n}$ it holds:

$$\begin{aligned}
\theta_1^{j-} = \theta_{j-}(k_{j-1,j}) &\geq 0 \\
\frac{p_h - p_j + tR - t(k_j - k_{j-1,j})}{q_h - q_l} &\geq 0 \\
\frac{p_h - p_j + tR - t\left(k_j - \frac{1}{2}\left(\frac{p_j - p_{j-1}}{t} + k_j + k_{j-1}\right)\right)}{q_h - q_l} &\geq 0 \\
2p_h - p_j &\geq p_{j-1} - 2tR + t(k_j - k_{j-1}) \\
2p_h - p_j &\geq p_{j-1} - 2tR + \frac{t}{n}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
\theta_2^j = \theta_{j-}(k_j) &\leq 1 \\
\frac{p_h - p_j + tR}{q_h - q_l} &\leq 1 \\
p_h - p_j &\leq q_h - q_l - tR
\end{aligned} \tag{20}$$

Considering deviating strategies of the railway operator, it is assumed that all bus operators set $p_j = p_l^N$. Then (19) and (20) reduce to

$$\theta_1^{j-} \geq 0 \quad \Leftrightarrow \quad p_h \geq p_l^N - tR + \frac{t}{2n} \tag{21}$$
$$\theta_2^{j} \leq 1 \quad \Leftrightarrow \quad p_h \leq p_l^N + q_h - q_l - tR \tag{22}$$

In the trapezoid case railway's demand in the segment between $k_{j-1,j}$ and k_j is (compare (3)):

$$\begin{aligned}
D_h(p_h) &= (k_j - k_{j-1,j})(1 - \theta_{j-}(k_j)) + \frac{1}{2}(k_j - k_{j-1,j})(\theta_{j-}(k_j) - \theta_{j-}(k_{j-1,j})) \\
&= (k_j - k_{j-1,j}) \left(1 - \frac{1}{2}\theta_{j-}(k_j) - \frac{1}{2}\theta_{j-}(k_{j-1,j})\right) \\
&= \frac{1}{2n}\left(1 - \frac{1}{2}\frac{p_h - p_l^N + tR}{q_h - q_l} - \frac{1}{2}\frac{p_h - p_l^N + tR - t\frac{1}{2n}}{q_h - q_l}\right)
\end{aligned}$$

Its derivation is:

$$D_h'(p_h) = -\frac{1}{2n(q_h - q_l)} \tag{23}$$

and therefore the second derivation is $D_h''(p_h) = 0$. Railway operator's profit is $\pi_h(p_h) = D_h(p_h)p_h$, and its first and second derivatives are:

$$\begin{aligned}
\pi_h'(p_h) &= D_h(p_h) + D_h'(p_h)p_h \\
\pi_h''(p_h) &= 2D_h'(p_h) + D_h''(p_h)p_h = 2D_h'(p_h) < 0
\end{aligned}$$

Hence, the railway operator's profit function is strictly concave in the trapezoid region. Since the first oder condition is satisfied at $p_h = p_h^N$ it follows that the railway operator will not deviate from p_h^N within the trapezoid region.

b) The rail operator sets a high price such that $\theta_2^j > 1$

If the rail operator raises its price strongly enough its demand (over the segment) will shrink to a triangle as shown in Figure 5 in light shade:

$$D_h(p_h) = \frac{1}{2}\left(1 - \theta_{j-}(k_{j-1,j})\right)\left(\hat{k} - k_{j-1,j}\right)$$

where $\hat{k}$ is given by the condition $\theta_{j-}(\hat{k}) = 1$, which solves for

$$\hat{k} = -\frac{p_h - p_l^N - (q_h - q_l) + tR - t\frac{j-1}{n}}{t}$$

Inserting everything yields

$$D_h(p_h) = \frac{1}{2}\left(1 - \frac{p_h - p_l^N + tR - t\frac{1}{2n}}{q_h - q_l}\right)\left(\frac{-p_h + p_l^N + q_h - q_l - tR + t\frac{j-1}{n}}{t} - \frac{2j-3}{2n}\right) \tag{24}$$

with derivation:

$$D_h'(p_h) = \frac{p_h - p_l^N - q_h + q_l + tR - \frac{t}{2n}}{t(q_h - q_l)} \tag{25}$$

From (22) the rail operator's price must be at least $p_h = p_l^N + q_h - q_l - tR$ in order to satisfiy $\theta_2^j \geq 1$. There is also a maximal value for p_h, namely, where demand drops to zero:

$$\theta_{j-}(k_{j-1,j}) = 1$$

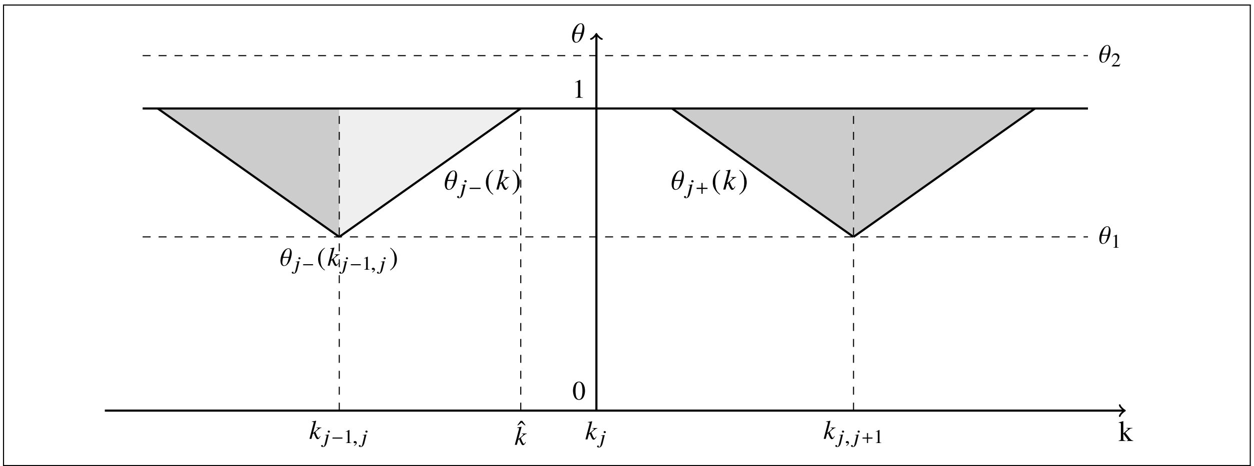

Figure 5: Market segments when the rail operator chooses a very high price

$$\frac{p_h - p_l^N + tR - \frac{t}{2n}}{q_h - q_l} = 1$$

$$p_h = p_l^N + q_h - q_l - tR + \frac{t}{2n} \tag{26}$$

Hence, we get the relevant interval $p_h \in [p_l^N + q_h - q_l - tR, p_l^N + q_h - q_l - tR + \frac{t}{2n}]$. We show that rail operator's profit $\pi_h(p_h) = D_h(p_h)p_h$ is decreasing in that interval.

The derivation of the profit function is generally:

$$\pi_h'(p_h) = D_h(p_h) + D_h^{'}(p_h)p_h \tag{27}$$

Clearly, the demand function is continuous at the point where it changes from the trapezoid to the triangle form. But even its derivative is continuous, since at the minimal value, $p_h = p_l^N + q_h - q_l - tR$, the derivation (25) reduces to $-\frac{1}{2n(q_h - q_l)}$, identical to (23). It follows that also the derivation of the profit function is continuous at the point where demand changes from the trapezoid to the triangle form. At that point the profit function is already decreasing in the trapezoid region (since parametric conditions assure that the equilibrium candidate is in the interior of the trapezoid region for every player). Hence it follows that at the beginning of the triangle region, close to the minimal value of p_h, the derivative of the profit function is negative.

Substituting (24) and (25) into (27) gives:

$$\pi_h'(p_h) = \frac{1}{8}\frac{(2p_h - 2p_l^N - 2q_h + 2q_l + 2tR - \frac{t}{n})(6p_h - 2p_l^N - 2q_h + 2q_l + 2tR - \frac{t}{n})}{t(q_h - q_l)}$$

This expression has exactly two roots. One of them is:

$$2p_h - 2p_l^N - 2q_h + 2q_l + 2tR - \frac{t}{n} = 0 \quad \Leftrightarrow \quad p_h = p_l^N + q_h - q_l - tR + \frac{t}{2n}$$

Comparing to (26), this happens to be the maximum value where demand drops to zero, and thus profit as well. Since the derivative has just two distinct roots and since profit is never negative, there must be a neighborhood of the maximum value of p_h where the derivative of the profit function is strictly negative.

Hence, with respect to the relevant interval $p_h \in [p_l^N + q_h - q_l - tR, p_l^N + q_h - q_l - tR + \frac{t}{2n}]$, we have shown that the derivative of the profit function is negative both at the minimum value and close to the maximum value of p_h. Since there is just one other root of the derivative, it follows that the derivative cannot be strictly positive anywhere in the relevant interval. So the profit function is strictly decreasing in that interval and,

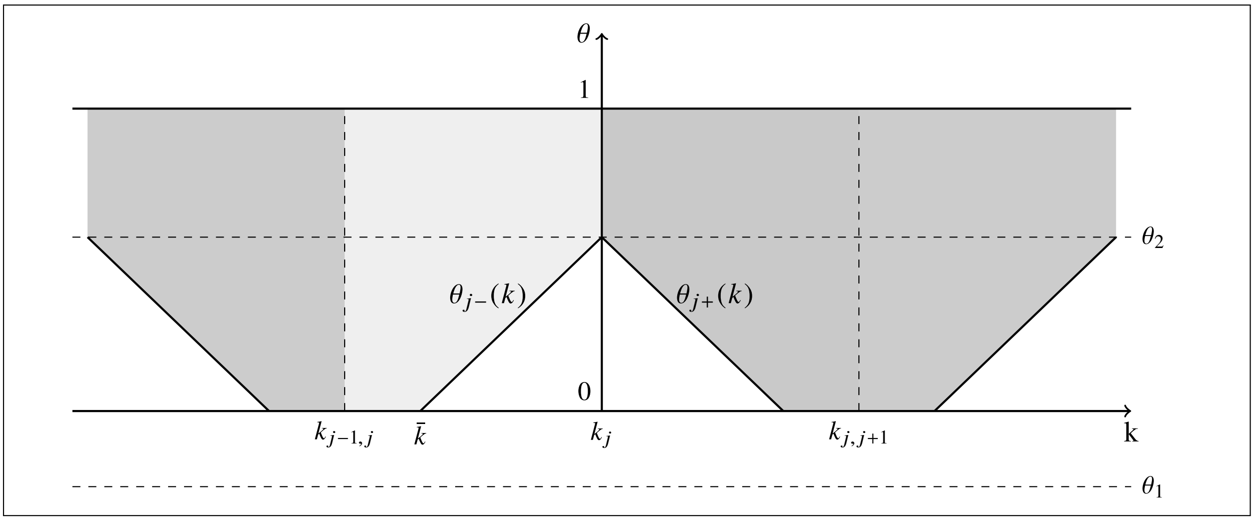

Figure 6: Market segments when the rail operator chooses a very low price

as a consequence, the railway operator will not deviate from the proposed equilibrium strategy by setting a higher price.

c) The rail operator sets a low price such that $\theta_1^{j-} < 0$.

If the rail operator reduces its price strongly its demand on the segment will grow to become the whole segment rectangle minus a triangle, as shown in Figure 6 in light shade:

$$D_h(p_h) = (k_j - k_{j-1,j}) - \frac{1}{2}\theta_{j-}(k_j)(k_j - \bar{k})$$

where $\bar{k}$ is given by the condition $\theta_{j-}(\bar{k}) = 0$, which solves for

$$\bar{k} = \frac{-p_h + p_l^N - tR + t\frac{j-1}{n}}{t} \tag{28}$$

Inserting everything gives

$$D_h(p_h) = \frac{1}{2n} - \frac{1}{2}\frac{(p_h - p_l^N + tR)^2}{t(q_h - q_l)}$$

with derivation:

$$D_h^{'}(p_h) = -\frac{p_h - p_l^N + tR}{t(q_h - q_l)} \tag{29}$$

According to (21), the maximum value of p_h, where the change from the trapezoid region to this region occurs, is $p_h = p_l^N - tR + \frac{t}{2n}$. The minimum value of p_h is reached when rail operator's demand is at its maximum, i.e. coincides with the whole rectangle of the segment:

$$\begin{aligned}\theta_{j-}(k_j) &= 0\\ \frac{p_h - p_l^N + tR}{q_h - q_l} &= 0\\ p_h &= p_l^N - tR\end{aligned}$$

With these bounds the relevant interval is $p_h \in [p_l^N - tR, p_l^N - tR + \frac{t}{2n}]$.

The derivative of the profit function is $\pi_h'(p_h) = D_h(p_h) + D_h'(p_h)p_h$. The demand expression $D_h(p_h)$ is of course decreasing. But so is the other term of $\pi_h'(p_h)$, since:

$$(D_h'(p_h)p_h)' = D_h'(p_h) + D_h''(p_h)p_h < D_h''(p_h)p_h = -\frac{p_h}{t(q_h - q_l)} < 0$$

As a consequence, the profit function is strictly concave over the relevant interval, i.e. $\pi_h''(p_h) < 0$.

Moreover, the derivative of the profit function is continuous at the maximum value of p_h, where the change from the trapezoid region to this region occurs. This is so because the demand function is of course continuous, but also its derivative, as expression (29) at $p_h = p_l^N - tR + t\frac{1}{2n}$ reduces to $-\frac{1}{2n(q_h - q_l)}$, identical to (23). At that point the profit function is increasing in the trapezoid region (since parametric conditions assure that the equilibrium candidate is in the interior of the trapezoid region for every player). Hence it follows that close the boundary of the region considered here, i.e. close to the maximal value of p_h, the profit function is also increasing.

Taken together, we have shown that the profit function is strictly concave over the relevant interval and still increasing at its maximal value. Hence it is strictly increasing over that whole interval, and, as a consequence, the railway operator will not deviate from the proposed equilibrium strategy by setting a lower price.

d) Bus operator j sets a price such that $0 \geq \theta_1^{j-}$ and $\theta_2^j \leq 1$ are preserved (trapezoid case)

Considering deviating strategies of bus operator j, it is assumed that the neighboring bus operator sets $p_{j-1} = p_l^N$ and the rail operator sets $p_h = p_h^N$. Then (19) and (20) reduce to

$$\theta_1^{j-} \geq 0 \quad \Leftrightarrow \quad p_j \leq 2p_h^N - p_l^N + 2tR - \frac{t}{n} \tag{30}$$

$$\theta_2^j \leq 1 \quad \Leftrightarrow \quad p_j \geq p_h^N - q_h + q_l + tR \tag{31}$$

We show that within the trapezoid region bus operator's profit is strictly quasi-concave with its maximum at the proposed equilibrium. In the trapezoid region the demand of the bus operator in the segment is (compare (2)):

$$\begin{aligned} D_j(p_j) &= (k_j - k_{j-1,j})(\theta_{j-}(k_{j-1,j})) + \frac{1}{2}(k_j - k_{j-1,j})(\theta_{j-}(k_j) - \theta_{j-}(k_{j-1,j})) \\ &= (k_j - k_{j-1,j})\left(\frac{1}{2}\theta_{j-}(k_j) + \frac{1}{2}\theta_{j-}(k_{j-1,j})\right) \\ &= (k_j - k_{j-1,j})\left(\frac{2p_h^N - 2p_j + 2tR - t(k_j - k_{j-1,j})}{2q_h - 2q_l}\right) \end{aligned} \tag{32}$$

where $k_{j-1,j}$ and k_j are given by expression (1) and $k_j = (j-1)/n$, respectively. Inserting these into (32) and setting $p_h = p_h^N$ and $p_{j-1} = p_l^N$ yields

$$D_j(p_j) = -\frac{1}{8}\frac{\left(p_j - p_l^N - \frac{t}{n}\right)\left(4p_h^N - 3p_j - p_l^N + 4tR - \frac{t}{n}\right)}{t(q_h - q_l)}$$

with derivation:

$$D_j'(p_j) = -\frac{1}{4}\frac{2p_h^N - 3p_j + p_l^N + 2tR + \frac{t}{n}}{t(q_h - q_l)} \tag{33}$$

Bus operator's profit is $\pi_j(p_j) = D_j(p_j)p_j$, and its first and second derivatives are:

$$\pi_j'(p_j) = -\frac{1}{8t(q_h - q_l)}\Big((p_l^N)^2 - 9p_j^2 + 4p_jp_l^N + 8p_h^Np_j - 4p_h^Np_l^N + 8p_jtR$$

$$
\begin{aligned}
&- 4p_l^N tR - 4p_h^N \frac{t}{n} + 4p_j \frac{t}{n} + 2p_l^N \frac{t}{n} - 4tR\frac{t}{n} + \frac{t^2}{n^2}\Big) \\
\pi_j''(p_j) &= - \frac{1}{4t(q_h - q_l)} \left(4p_h^N - 9p_j + 2p_l^N + 4tR + 2\frac{t}{n}\right)
\end{aligned}
$$

The first derivative of the profit has two real roots given by:

$$
\begin{aligned}
p_{j\pm} =& \frac{4}{9}p_h^N + \frac{2}{9}p_l^N + \frac{4}{9}tR + \frac{2}{9}\frac{t}{n} \\
& \pm \Big(16(p_h^N)^2 - 20p_h^N p_l^N + 13(p_l^N)^2 + 32p_h^N tR - 20p_l^N tR \\
& \quad -20p_h^N \frac{t}{n} + 26p_l^N \frac{t}{n} + 16t^2R^2 - 20tR\frac{t}{n} + 13\frac{t^2}{n^2}\Big)^{\frac{1}{2}}
\end{aligned}
$$

It follows that $\pi_j''(p_{j-}) < 0$ and $\pi_j''(p_{j+}) > 0$, implying that the smaller root is a local maximum and the larger one a local minimum. In fact, plugging the solution for p_l^N and p_h^N into the smaller root yields $p_{j-} = p_l^N$, so that the smaller root is our equilibrium candidate.

Note that by imposing conditions (9) and (10) we assured that the equilibrium candidate satisfies the boundary conditions (30) and (31) with strict inequalities, i.e. the smaller root is strictly within the trapezoid region. Consequently, the profit function is strictly increasing (in this region) up to the smaller root and strictly decreasing for some range after it. We now show that at the upper bound of the trapezoid region it is still strictly decreasing, implying strict quasi-concavity of the profit function over the whole trapezoid region.

At the upper bound of the trapezoid region it holds $p_j = 2p_h^N - p_l^N + 2tR - \frac{t}{n}$, by (30). At this point the derivative of the profit function is

$$
\begin{aligned}
\pi_j'\left(2p_h^N - p_l^N + 2tR - \frac{t}{n}\right) &= \frac{1}{2}\frac{\left(p_h^N - p_l^N + tR - \frac{t}{n}\right)\left(5p_h^N - 3p_l^N + 5tR - 3\frac{t}{n}\right)}{t(q_h - q_l)} \\
&= \frac{1}{2}D_j'\left(2p_h^N - p_l^N + 2tR - \frac{t}{n}\right)\left(5p_h^N - 3p_l^N + 5tR - 3\frac{t}{n}\right)
\end{aligned}
$$

where (33) has been used. Since it holds $D_j'(p_j) < 0$, it remains to be shown that $5p_h^N - 3p_l^N + 5tR - 3\frac{t}{n} > 0$. From condition (9) in the form $p_h^N - p_l^N > -tR + \frac{t}{2n}$ it follows:

$$
5p_h^N - 3p_l^N + 5tR - 3\frac{t}{n} > 2p_l^N - \frac{t}{2n}
$$

And from condition (9) in the form $A > \frac{t}{n}$ it follows that $2p_l^N - \frac{t}{2n} > 0$. More explicitly:

$$
2p_l^N - \frac{1}{2n} > 0 \quad \Leftrightarrow \quad A + \frac{7}{4}\frac{t}{n} - \frac{1}{4}\sqrt{B} > 0 \quad \Leftrightarrow \quad \sqrt{B} < 4A + 7\frac{t}{n} \quad \Leftrightarrow \quad \frac{t}{n} < A
$$

e) Bus operator j sets a high price such that $\theta_1^{j-} < 0$

If the bus operator raises its price strongly enough its demand will shrink to a triangle as shown in Figure 7 in light shade:

$$
D_j(p_j) = \frac{1}{2}\left(k_j - \bar{k}\right)\theta_-(k_j)
$$

where $\bar{k}$ is given by the equation (28). Inserting everything yields

$$
D_j(p_j) = \frac{1}{2}\frac{\left(p_h^N - p_j + tR\right)^2}{t(q_h - q_l)} \tag{34}
$$

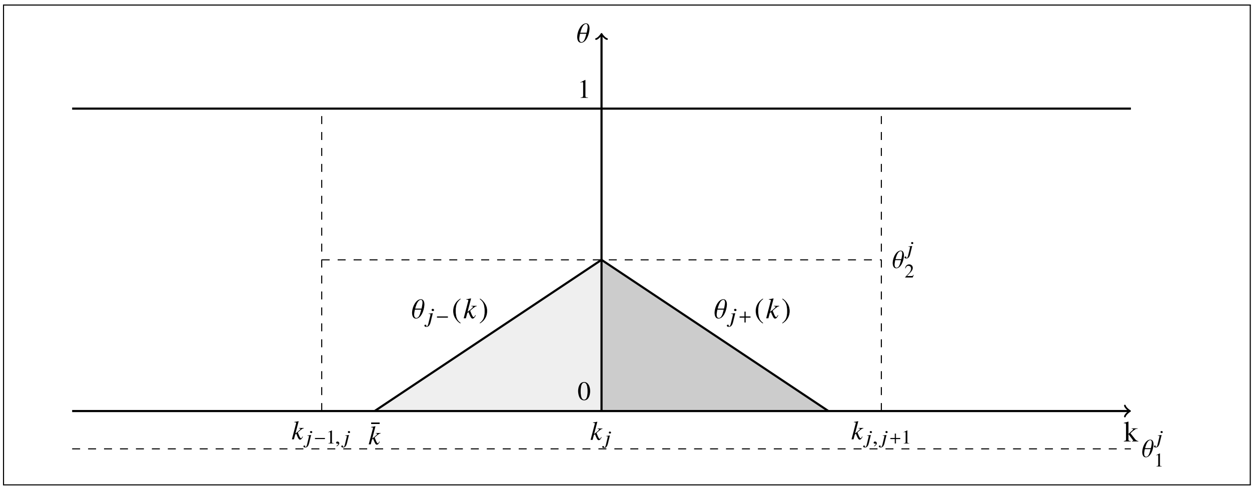

Figure 7: Market segments when the bus operator chooses a very high price

with derivation:

$$D_j'(p_j) = -\frac{p_h^N - p_j + tR}{t(q_h - q_l)} \tag{35}$$

The point where the trapezoid becomes a triangle is given by (30) as $p_j = 2p_h^N - p_l^N + 2tR - \frac{t}{n}$. This is therefore the minimal value for p_j for this triangle region. There is also a maximal value for p_j, namely, where demand drops to zero:

$$\theta_{j-}(k_j) = 0 \quad \Leftrightarrow \quad \frac{p_h^N - p_j + tR}{q_h - q_l} = 0 \quad \Leftrightarrow \quad p_j = p_h^N + tR \tag{36}$$

Hence we get the relevant interval $p_j \in [2p_h^N - p_l^N + 2tR - \frac{t}{n}, p_h^N + tR]$. Clearly, the demand function is continuous at the boundary where it changes from the trapezoid to the triangle region. But even its derivative is continuous, since at the boundary the derivations (33) and (35) both reduce to the same expression:

$$D_j'\left(2p_h^N - p_l^N + 2tR - \frac{t}{n}\right) = \frac{p_h^N - p_l^N + tR - \frac{t}{n}}{t(q_h - q_l)}$$

The bus operator's profit is $\pi_j(p_j) = D_j(p_j)p_j$, with derivation $\pi_j'(p_j) = D_j(p_j) + D_j'(p_j)p_j$. Since demand and derivative of demand are continous at the boundary between the trapezoid and the triangle regions, the same must hold for profit and derivative of profit. In the analysis of the trapezoid case – i.e. case d) of this proof – it was shown that the derivative of the profit function is negative at this boundary. Hence it follows that at the beginning of the triangle region, the derivative of the profit function is negative.

Substituting (34) and (35) gives:

$$\pi_j'(p_j) = D_j(p_j) + D_j'(p_j)p_j = \frac{1}{2}\frac{(p_h^N - p_j + tR)(p_h^N - 3p_j + tR)}{t(q_h - q_l)} \tag{37}$$

This expression has exactly two roots. One of them is:

$$p_h^N - p_j + tR = 0 \quad \Leftrightarrow \quad p_j = p_h^N + tR \tag{38}$$

Comparing to (36), this happens to be the maximum value where demand drops to zero, and thus profit as well. Since the derivation has just two distinct roots and since profit is never negative, there must be a neighborhood of the maximum value of p_j where the derivative of the profit function is strictly negative.

Hence, with respect to the relevant interval $p_j \in [2p_h^N - p_l^N + 2tR - \frac{t}{n}, p_h^N + tR]$, we have shown that the derivative of the profit function is negative both at the minimum value and close to the maximum value of p_j. Since there is just one other root of the derivative, it follows that the derivative cannot be strictly positive anywhere in the relevant interval. So the profit function is strictly decreasing in that interval and, as a consequence, a bus operator j will not deviate from the proposed equilibrium strategy by setting a higher price.

f) Bus operator j sets a low price such that $\theta_2^j > 1$

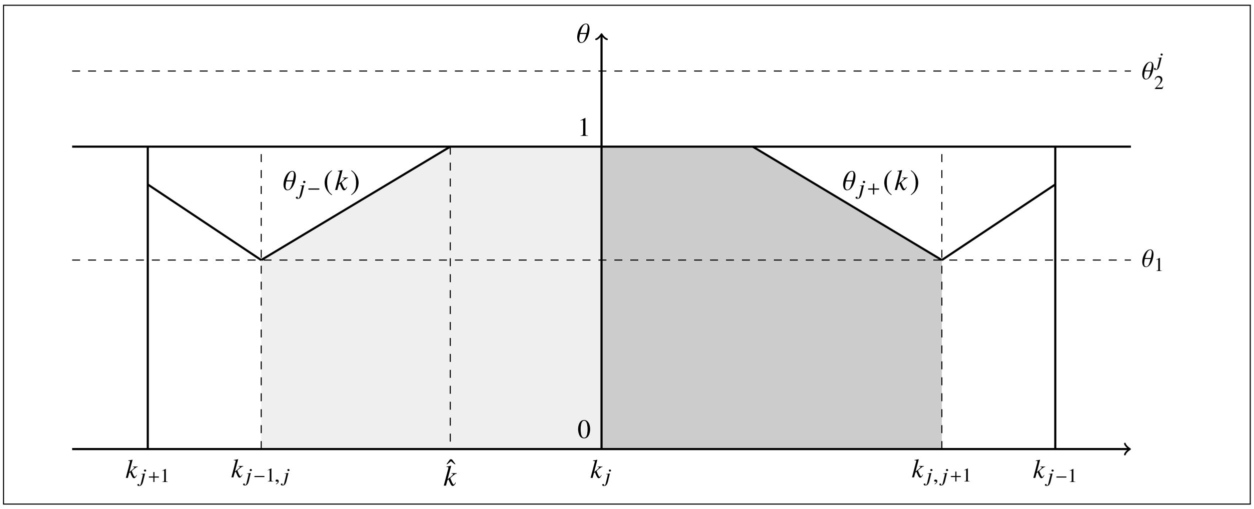

Figure 8: Market segments when the bus operator chooses a very low price

If bus operator j chooses a very low price, then $\theta_2^j > 1$ and its demand becomes the rectangle $1 \cdot (k_j - k_{j-1,j})$ less the triangele $\frac{1}{2}(\hat{k} - k_{j-1,j})(1 - \theta_{j-}(k_{j-1,j}))$, as shown in Figure 8. We show that in this case the derivation of the profit function is continuous and increasing throughout.

The critical point where the regime change occurs characterizes the maximal value of p_j for this region. By (31) it is given by $p_j = p_h^N - q_h + q_l + tR$. The minimal value of p_j for this region is given by the condition that $k_{j-1,j}$ is equal to k_{j-1}:

$$\begin{aligned} k_{j-1,j} &= k_{j-1} \\ \frac{1}{2}\left(\frac{p_j - p_{j-1}}{t} + k_j + k_{j-1}\right) &= k_{j-1} \\ p_j &= p_{j-1} - \frac{t}{n} \;=\; p_l^N - \frac{t}{n} \end{aligned}$$

Thus, the relevant interval for this region is $p_j \in [p_l^N - \frac{t}{n}, p_h^N - qh + ql + tR]$. Following Figure 8, in the region considered here the demand is given by:

$$D_j(p_j) = (k_j - k_{j-1,j}) - \frac{1}{2}(\hat{k} - k_{j-1,j})\left(1 - \theta_{-}(k_{j-1,j})\right)$$

which leads to a fairly long expression omitted here and to the derivation:

$$D_j^{'}(p_j) = \frac{1}{4}\frac{2p_h^N - p_j - p_l^N - 4q_h + 4q_l + 2tR - \frac{t}{n}}{t(q_h - ql)} \tag{39}$$

The derivation of the profit function is given by $\pi_j'(p_j) = D_j(p_j) + D_j^{'}(p_j)p_j$. The demand expression $D_j(p_j)$ is of course decreasing. But so is the other term of $\pi_j^{'}(p_j)$, since:

$$(D_j^{'}(p_j)p_j)^{'} = D_j^{'}(p_j) + D_j^{''}(p_j)p_j < D_j^{''}(p_j)p_j = -\frac{1}{4}\frac{p_j}{t(q_h - q_l)}$$

Hence it follows that $\pi_j''(p_j) < 0$, i.e. the profit function is strictly concave over this whole region.

Moreover, the derivative of the profit function is continuous at the maximum value of p_j, where the change from this region to the trapezoid region occurs. This is so because at the boundary the derivations (33) and (39) both reduce to the same expression:

$$D_j'\left(p_h^N - q_h + q_l + tR\right) = \frac{1}{4}\frac{p_h - p_l^N - 3(q_h - q_l) + tR - \frac{t}{n}}{t(q_h - q_l)}$$

At that point the profit function is increasing in the trapezoid region; see part d) of the proof. Hence, it follows that close to the boundary of the region considered here, i.e. close to the maximal value of p_j, the profit function is also increasing.

Taken together, we have shown that the profit function is strictly concave over the relevant interval and still increasing at its maximal value. Hence, it is strictly increasing over that whole interval, and, as a consequence, the bus operator j will not deviate from the proposed equilibrium strategy by setting such a low price.

However, it might set an even lower price, namely, lower than the lower bound considered so far: $p_j < p_l^N - \frac{t}{n}$. In that case there will be a discrete jump in demand (and thus profit), since operator j will then at once capture the whole demand so far served by its two neighboring bus operators. However, it holds that $p_l^N < \frac{t}{n}$:

$$p_l^N < \frac{t}{n} \quad \Leftrightarrow \quad \frac{1}{2}\left(A + \frac{9}{4}\frac{t}{n}\right) - \frac{1}{8}\sqrt{B} < \frac{t}{n} \quad \Leftrightarrow \quad 4A + \frac{t}{n} < \sqrt{B} \quad \Leftrightarrow \quad 16A^2 + 8A\frac{t}{n} + \left(\frac{t}{n}\right)^2 < B$$

By definition of B this is always satisfied. As a consequence, price and profit are already negative at $p_j = p_l^N - \frac{t}{n}$ and the upward jump in demand would lead to a *downward* jump of profit, making it even more negative. Clearly, a bus operator would never do this.

A.4 Expressions for checking equilibrium in the simplified model and numerical example

In order to check whether a given numerical set of parameters yields the candidate equilibrium as a Nash equilibrium of the game one has to consider the whole strategy spaces of the individual players. Given that all other players play the strategy of the candidate equilibrium, the player unter consideration may set its price in such a way that the generic constellation of demand is not valid. As a reminder, by the *generic constellation* of demand we refer to a situation in which all the consumers with θ_3 will take the railways, consumers with θ_2 will partly choose the bus and partly the railways, and consumers with θ_1 will choose between buses only. It has been shown in section 4.2 that no player has an incentive to deviate from the equilibrium strategy as long as the generic constellation is kept. So it remains to be checked whether he has no incentive to choose a price such that the generic constellation breaks up. This has to be checked for the railway company and for one of the symmetric bus companies.

Railway company. Let $z \in \{1,2,3\}$ and $I_z = \left(p_l^N + \theta_z(q_h - q_l) - tR,\ p_l^N + \frac{t}{2n} + \theta_z(q_h - q_l) - tR\right)$. We show that railway's demand on the θ_z-level is:

$$D_h^z(p_h, p_l^N) = \begin{cases} \lambda_z & \text{if } p_h \leq p_l^N + \theta_z(q_h - q_l) - tR \\ \lambda_z - \frac{\lambda_z}{t}\left(p_h - p_l^N + tR - \theta_z(q_h - q_l)\right) & \text{if } p_h \in I_z \\ 0 & \text{if } p_h \geq p_l^N + \frac{t}{2n} + \theta_z(q_h - q_l) - tR \end{cases}$$

The notation $D_h^z(p_h, p_l^N)$ shall just remind us that all buses choose p_l^N and only the railway price p_h is free to change. If $p_h \leq p_l^N + \theta_z(q_h - q_l) - tR$ then the railway gets the full demand of the θ_z-level, so that $D_h^z(p_h, p_l^N) = \lambda_z$. This is so because even the consumer who lives at the position of a bus company (i.e. at some k_j) would get a utility from bus ride ($V_0 + \theta_z q_l - p_l^N$) which is lower than the utility from using the railways ($V_0 + \theta_z q_h - p_h - tR$). On the other extreme, if $p_h \geq p_l^N + \frac{t}{2n} + \theta_z(q_h - q_l) - tR$ then the railway gets no demand at all at this θ_z-level. This is so because even the consumer who lives exactly between two buses (i.e. at some $k_{j,j+1}$) gets a utility from bus ride ($V_0 + \theta_z q_l - p_l^N - t|k_{j,j+1} - k_j|$) which is higher than the utility from using the railways ($V_0 + \theta_z q_h - p_h - tR$).

Finally, for $p_h \in I_z$, railway demand on the θ_z-level is very similar as stated and explained in the main text for the θ_2-level, namely,

$$\begin{aligned}
D_h^z(p_h, p_l^N) &= \lambda_z \cdot \sum_{j=1}^{n} \left[(k_{j-} - k_{j-1,j}) + (k_{j,j+1} - k_{j+}) \right] \\
&= 2n\lambda_z (k_{j+1,j} - k_{j+}) \\
&= \lambda_z - \frac{\lambda_z}{t} \left(p_h - p_l^N + tR - \theta_z(q_h - q_l) \right)
\end{aligned}$$

where the symmetry has been used in the second line and (11) (with θ_2 replaced by θ_z) has been used in the third line. Total demand of the railway company is the sum of its demands over all three levels:

$$D_h(p_h, p_l^N) = D_h^1(p_h, p_l^N) + D_h^2(p_h, p_l^N) + D_h^3(p_h, p_l^N)$$

And its profit is $\pi_h(p_h, p_l^N) = p_h D_h(p_h, p_l^N)$.

Bus company j**.** We consider a segment (k_j, k_{j+1}) between two bus operators on θ_z-level. Similar to (11), the consumer who is indifferent between the rail and bus j is given by $k_{j+}^z = \frac{1}{t}\left(p_h^N - p_j + tR - \theta_z(q_h - q_l) \right) + k_j$. Likewise, the consumer who is indifferent between the rail and bus $j+1$ (which sets p_l^N) is given by $k_{j+1-}^z = \frac{1}{t}\left(\theta_z(q_h - q_l) + p_l^N - p_h^N - tR \right) + k_{j+1}$. Therefore, bus j is directly competing with the rail if $k_{j+}^z < k_{j+1-}^z$, which is equivalent to

$$p_j > 2p_h^N - p_l^N + 2tR - 2\theta_z(q_h - q_l) - \frac{t}{n}$$

If this holds, the resulting demand of bus j on θ_z-level is $\max\{2\lambda_z(k_{j+}^z - k_j), 0\}$ where $k_{j+}^z - k_j = \frac{1}{t}(p_h^N - p_j + tR - \theta_h(q_h - q_l)$. We have thus shown the lower part of the following expression of bus j's demand:

$$D_j^z(p_j, p_l^N, p_h^N) = \begin{cases} \max\left\{ \lambda_z \left(\frac{1}{t}(p_l^N - p_j) + \frac{1}{n} \right), 0 \right\} \\ \qquad \text{if } p_j < 2p_h^N - p_l^N + 2tR - 2\theta_z(q_h - q_l) - \frac{t}{n} \\ \max\left\{ \frac{2}{t}\lambda_z \left(p_h^N - p_j + tR - \theta_z(q_h - q_l) \right), 0 \right\} \\ \qquad \text{if } p_j \geq 2p_h^N - p_l^N + 2tR - 2\theta_z(q_h - q_l) - \frac{t}{n} \end{cases}$$

For the upper part: If p_j is set below the critical expression, bus j is directly competing with bus $j+1$ on level θ_z. The critical consumer between the two buses is denoted by $k_{j,j+1}^z$, and the resulting demand of bus j is $\max\{2\lambda_z(k_{j,j+1}^z - k_j), 0\}$. Using (1) and adapting, $k_{j,j+1}^z - k_j = \frac{1}{2t}(p_l^N - p_j) + \frac{1}{2n}$.

Total demand of bus company j is the sum of its demands over all three levels:

$$D_j(p_j, p_l^N, p_h^N) = D_j^1(p_j, p_l^N, p_h^N) + D_j^2(p_j, p_l^N, p_h^N) + D_j^3(p_j, p_l^N, p_h^N)$$

And its profit is $\pi_j(p_j, p_l^N, p_h^N) = p_j D_j(p_j, p_l^N, p_h^N)$.

As in the general model, one also has to consider the possibility that the bus operator sets a very low price, such that it captures the full demand of the neighbor bus, implying a discret jump in its demand. As in the general model, this happens when $p_j < p_l^N - \frac{t}{n}$. However, we now show that $p_l^N < \frac{t}{n}$, so that the bus operator would make a negative profit.

Assume by contradiction that $p_l^N \geq \frac{t}{n}$. Inserting the expression for p_l^N it can be rearranged:

$$\theta_2(q_h - q_l) - tR \leq \frac{\frac{1}{2} - \frac{1}{2}\lambda_1 - 3\lambda_2}{\lambda_2} \frac{t}{n}$$

Using condition (15) as well as the expressions for p_l^N and p_h^N:

$$-\frac{t}{2n}\left[\frac{\lambda_1^2+4\lambda_1\lambda_2-\lambda_1-2\lambda_2+2\lambda_2(3\lambda_2+\lambda_1)}{(2\lambda_2+\lambda_1)\lambda_2}\right]<\theta_2(q_h-q_l)-tR$$

Therefore, the following equation should be satisfied:

$$-\frac{t}{2n}\left[\frac{\lambda_1^2+4\lambda_1\lambda_2-\lambda_1-2\lambda_2+2\lambda_2(3\lambda_2+\lambda_1)}{(2\lambda_2+\lambda_1)\lambda_2}\right]<\frac{\frac{1}{2}-\frac{1}{2}\lambda_1-3\lambda_2}{\lambda_2}\frac{t}{n}$$

$$\Leftrightarrow 2\lambda_1\lambda_2+6\lambda_2^2<0$$

This is in contradiction to the assumption $\lambda_i > 0$.

Numerical Example. Table 4 lists the parameters which we chose as numerical example, where n has been set exogeneously, and the resulting values of the candidate price equilibrium.

Parameters	$q_l = 0$	$q_h = 0.3$	
	$t = 1.2$	$R = 0.1$	$n = 4$
	$\theta_1 = 0$	$\theta_2 = 0.5$	$\theta_3 = 1$
	$\lambda_1 = 0.2$	$\lambda_2 = 0.6$	$\lambda_3 = 0.2$
Equilibrium	$p_h^N = 0.1555$	$x_h^N = 0.6220$	$\pi_h^N = 0.0967$
	$p_l^N = 0.0810$	$x_l^N = 0.0945$	$\pi_l^N = 0.0077$

Table 4: Numerical example for the simplified model

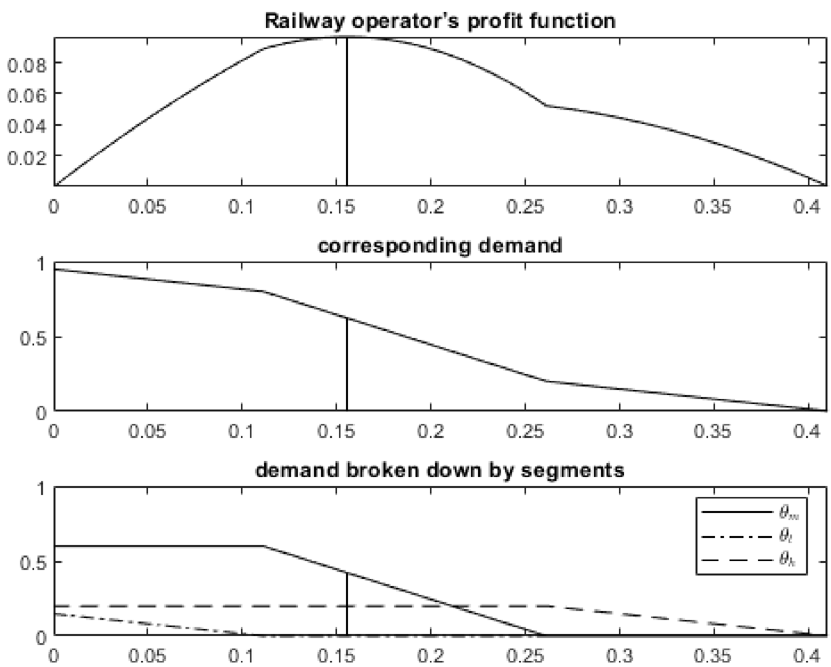

Figure 9: Demand and profit of the railway company in the numerical example

The example satisfies the conditions (12) to (15). Moreover, the Figures 9 and 10 demonstrate that the profits of the railway company and of a bus company are maximized by the respective candidate equilibrium

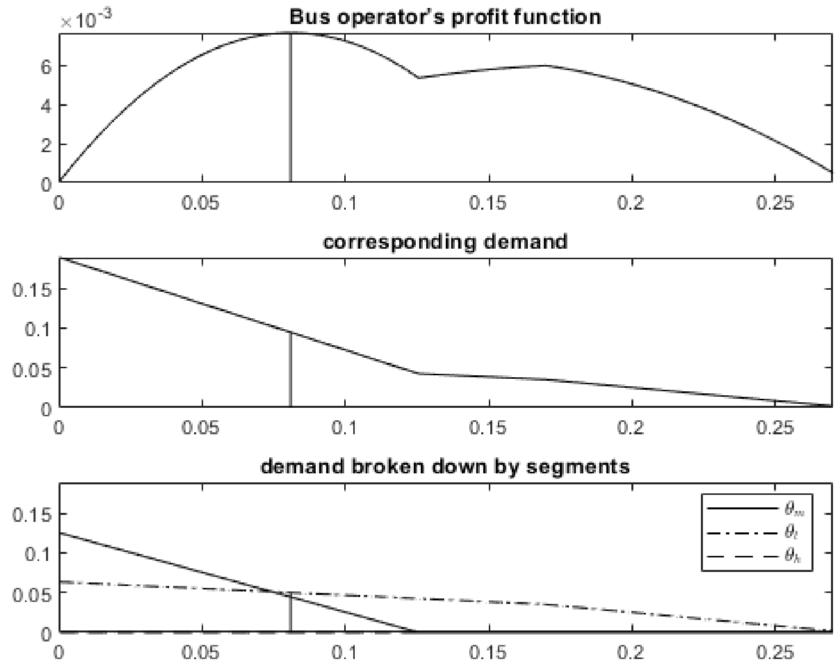

Figure 10: Demand and profit of a bus company in the numerical example

prices, which are highlighted by vertical lines, considering all the possibilities of demand besides the generic constellation. Hence, the candidate equilibrium is in fact a Nash equilibrium. The demand functions are also shown, both, total demand and demand on each θ-level. The kinks in the demand functions indicate regime changes, where the generic constellation is located around the equilibrium prices.

Finally, we have also checked the same set of figures for integer variations of n. It turns out that the equilibrium candidate remains the Nash equilibrium for $n = 4$ to $n = 8$. However, for $n = 9$ the bus would have an incentive to deviate away from the generic constellation. Calculating back from the condition $\pi_l(n^N) = F$, using (16), the interval $n \in [4, 8]$ corresponds to the interval $F \in [0.0015, 0.0077]$ in terms of the entry cost.

A.5 Bus operator's profit function in the simplified model

The profit function of a bus operator in the first stage $\pi_l(n)$ is strictly monotonically decreasing if (and only if) its derivative is strictly negative.

$$\frac{\partial \pi_l(n)}{\partial n} = -\frac{\lambda_1 + 2\lambda_2}{(\lambda_1 + 3\lambda_2)^2 n^2}\left(\lambda_2\left(tR - \theta_2\left(q_h - q_l\right)\right) + \frac{t\left(\lambda_1 + 1\right)}{2n}\right)\left(\lambda_1 + 1\right) \overset{!}{<} 0$$

$$\Leftrightarrow \quad tR - \theta_2\left(q_h - q_l\right) > -\frac{t}{2n}\frac{\lambda_1 + 1}{\lambda_2} \tag{40}$$

At the end of section 4.2, it was established that for a valid choice of parameters in this model the condition (14) must hold. After substituting the expressions for the equilibrium prices, it can be rearranged.

$$\theta_2(q_h - q_l) - tR < p_h^N - p_l^N$$

$$\Leftrightarrow \quad -\left(\theta_2(q_h - q_l) - tR\right) > \frac{\lambda_1^2 + 4\lambda_1\lambda_2 - \lambda_1 - 2\lambda_2}{\lambda_2\left(2\lambda_2 + \lambda_1\right)}\frac{t}{2n} \tag{41}$$

In order to proof the assertion $\frac{\partial \pi_l(n)}{\partial n} < 0$, we will show that the right-hand side of inequality (41) is greater than or equal to the right-hand side of inequality (40).

$$
\begin{aligned}
& \frac{\lambda_1^2 + 4\,\lambda_1\lambda_2 - \lambda_1 - 2\,\lambda_2}{\lambda_2\,(2\,\lambda_2 + \lambda_1)}\,\frac{t}{2\,n} \geq -\frac{t}{2\,n}\,\frac{\lambda_1 + 1}{\lambda_2} \\
\Leftrightarrow \quad & \lambda_1^2 + 4\,\lambda_1\lambda_2 - \lambda_1 - 2\,\lambda_2 \geq -(\lambda_1 + 1)\,(2\,\lambda_2 + \lambda_1) = -\lambda_1^2 - 2\,\lambda_1\,\lambda_2 - \lambda_1 - 2\,\lambda_2 \\
\Leftrightarrow \quad & 2\,\lambda_1^2 + 6\,\lambda_1\,\lambda_2 \geq 0
\end{aligned}
$$

Since this last inequality holds per definition, the assertion is shown.

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